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Modeling Safety of Multistate Ageing Systems

Keywords

Modelling, ageing systems, multistate, safety

Abstract

First, basic notions of the multistate system safety analysis are introduced, i.e. the multistate components and the multistate system, the multistate system component safety function, the multistate system safety and the multistate system risk function are defined. Moreover, the multistate system component and the multistate system main safety characteristics, i.e. their mean values of the lifetimes and in the safety state subsets and in the particular safety states and standard deviations and the moment when the system risk function exceeds a fixed permitted level are determined. Furthermore, there are constructed safety models of the multi-state homogeneous and non-homogeneous series, parallel, "*m* out of *n*", consecutive "*m* out of *n*: F", series-parallel, parallel-series, series-"*m* out of *n*", "*m* out of *n*"-series, series-consecutive "*m* out of *n*: F" and consecutive "*m* out of *n*: F"-series systems and their safety functions are determined. Moreover, a very often met in practiceseries system composed of multistate subsystems identical with the considered earlier multistate systems is considered and its safety function is determined.

1. Introduction

Taking into account the importance of the safety and operating process effectiveness of real technical systems it seems reasonable to expand the two-state approach [Kołowrocki, 2004, 2008, 2014]to multi-state approach [Amari, 1997], [Aven, 1985, 1999, 1993], [Barlow, Wu, 1978], [Brunelle, Kapur, 1999], [Hudson, Kapur, 1982, 1985], [Lisnianski, Levitin, 2003], [Natvig, 1982], [Ohio, Nishida, 1984], [Hue, 1985], [Xue, Yang, 1995a,b], [Yu et al 1994], [Kołowrocki, Soszyńska-Budny, 2011] in safety analysis. The assumption that the systems are composed of multi-state components with safety states degrading in time [Guze, Kolowrocki, 2008], [Kołowrocki, 2004, 2014], [Kołowrocki, Soszyńska-Budny, 2011], [Xue, 1985], [Xue, Yang 1995 a, b] gives the possibility for more precise analysis of their safety and operational processes' effectiveness. This assumption allows us to distinguish a system safety critical state to exceed which is either dangerous for the environment or does not assure the necessary level of its operation process effectiveness. Then, an important system safety characteristic is the time to the moment of exceeding the system safety critical state and its distribution, which is called the system risk function. This distribution is strictly related to the system safety function that are basic characteristics of the multi-state system. The safety models of the considered here typical multistate system structures can be applied in the safety analysis of real complex technical systems. They may be successfully applied, for instance, to safety analysis, identification, prediction and optimization of the critical infrastructures.

First, basic notions of the multistate system safety analysis are introduced, i.e. the multistate components and the multistate system, the multistate system component safety function, the multistate system safety and the multistate system risk function are defined. Moreover, the multistate system component and the multistate system main safety characteristics, i.e. their mean values of the lifetimes and in the safety state subsets and in the particular safety states and standard deviations and the moment when the system risk function exceeds a fixed permitted level are determined.

Further, there are constructed safety models of the multi-state homogeneous and non-homogeneous series, parallel, "*m* out of *n*", consecutive "*m* out of *n*: F", series-parallel, parallel-series, series-"*m* out of *n*", "*m* out of *n*"-series, series-consecutive "*m* out of *n*: F" and consecutive "*m* out of *n*: F"-series systems and their safety functions are determined. Moreover, a very often met in practiceseries system composed of multistate subsystems identical with the considered earlier multistate systems is considered and its safety function is determined.

Moreover, the multistate systems safety analysis in the case of their components and subsystems dependencies is presented [Blokus-Roszkowska, Kołowrocki, 2014a,b]. Namely, in the considered multistate ageing system it is assumed that after changing the safety state by any of its components, the inside interactions among the remaining components may cause the change of those components' safety states.

The considered models applications to real technical systems safety prediction are illustrated.

2. Modelling Safety of Multistate Systems

In the multistate safety analysis to define the system with degrading components, we assume that:

– n is the number of the system components,
– E_i, $i = 1,2,...,n$, are components of a system,
– all components and a system under consideration have the safety state set $\{0,1,...,z\}$, $z \geq 1$,
– the safety states are ordered, the safety state 0 is the worst and the safety state z is the best,
– $T_i(u)$, $i = 1,2,...,n$, are independent random variables representing the lifetimes of components E_i in the safety state subset $\{u,u+1,...,z\}$, while they were in the safety state z at the moment $t = 0$,
– $T(u)$ is a random variable representing the lifetime of a system in the safety state subset $\{u,u+1,...,z\}$ while it was in the safety state z at the moment $t = 0$,
– the system states degrades with time t,
– $s_i(t)$ is a component E_i safety state at the moment t, $t \in <0,\infty)$, given that it was in the safety state z at the moment $t = 0$,
– $s(t)$ is a system S safety state at the moment t, $t \in <0,\infty)$, given that it was in the safety state z at the moment $t = 0$.

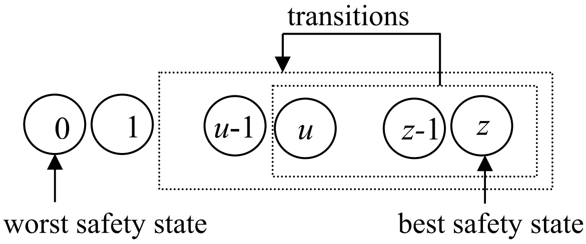

Figure 1. Illustration of a system and components safety states changing

The above assumptions mean that the safety states of the system with degrading components may be changed in time only from better to worse [Guze, Kołowrocki, 2008], [Kołowrocki 2004, 2014], [Kołowrocki, Soszyńska-Budny, 2011], [Xue, 1985], [Xue, Yang 1995 a, b]. The way in which the components and the system safety states change is illustrated in Figure 1.

Definition 1
A vector

$$S_i(t,\cdot) = [S_i(t,0),S_i(t,1),...,S_i(t,z)],\ t \in <0,\infty),\ i = 1,2,...,n, \quad (1)$$

where

$$S_i(t,u) = P(s_i(t) \geq u \mid s_i(0) = z) = P(T_i(u) > t),\ t \in <0,\infty),\ u = 0,1,...,z, \quad (2)$$

is the probability that the component E_i is in the safety state subset $\{u,u+1,...,z\}$ at the moment t, $t \in <0,\infty)$, while it was in the safety state z at the moment $t = 0$, is called the safety function of a multistate component E_i.

The safety functions $S_i(t,u)$, $t \in <0,\infty)$, $u = 0,1,...,z$, defined by (2) are called the coordinates of the component E_i, $i = 1,2,...,n$, safety function $S_i(t,\cdot)$ given by (1). Thus, the relationship between the distribution function $F_i(t,u)$ of the component E_i, $i = 1,2,...,n$, lifetime $T_i(u)$ in the safety state subset $\{u,u+1,...,z\}$ and the coordinate $S_i(t,u)$ of its safety function is given by

$$F_i(t,u) = P(T_i(u) \leq t) = 1 - P(T_i(u) > t) = 1 - S_i(t,u),\ t \in <0,\infty),\ u = 0,1,...,z.$$

Under *Definition 1* and the agreements, we have the following property of the multistate component safety function coordinates

$$S_i(t,0) \geq S_i(t,1) \geq \ldots \geq S_i(t,z),\ t \in <0,\infty),$$

$i = 1,2,...,n.$

Further, if we denote by

$$p_i(t,u) = P(s_i(t) = u \mid s_i(0) = z),\ t \in < 0,\infty),$$
$$u = 0,1,...,z,$$

the probability that the component E_i is in the safety state u at the moment t, while it was in the safety state z at the moment $t = 0$, then by (1)

$$S_i(t,0) = 1,\ S_i(t,z) = p_i(t,z),\ t \in < 0,\infty),$$
$$i = 1,2,...,n, \quad (3)$$

and

$$p_i(t,u) = S_i(t,u) - S_i(t,u+1),\ u = 0,1,...,z-1,$$
$$t \in < 0,\infty),\ i = 1,2,...,n. \quad (4)$$

Moreover, if

$$S_i(t,u) = 1 \text{ for } t \le 0,\ u = 1,2,...,z,\ i = 1,2,...,n,$$

then

$$\mu_i(u) = \int_0^\infty S_i(t,u)dt,\ u = 1,2,...,z,\ \ i=1,2,...,n, \quad (5)$$

is the mean lifetime of the component E_i in the safety state subset $\{u, u+1,..., z\}$,

$$\sigma_i(u) = \sqrt{n_i(u) - [\mu_i(u)]^2}\ ,$$
$$u = 1,2,...,z,\ i = 1,2,...,n, \quad (6)$$

where

$$n_i(u) = 2\int_0^\infty t S_i(t,u)dt\ ,\ u = 1,2,...,z,\ i = 1,2,...,n, \quad (7)$$

is the standard deviation of the component E_i lifetime in the safety state subset $\{u, u+1,..., z\}$, and

$$\bar{\mu}_i(u) = \int_0^\infty p_i(t,u)dt,\ u = 1,2,...,z,\ i = 1,2,...,n, \quad (8)$$

is the mean lifetime of the component E_i in the safety state u, in the case when the integrals defined by (5), (7) and (8) are convergent.
Next, according to (3), (4), (5) and (8), we have

$$\bar{\mu}_i(u) = \mu_i(u) - \mu_i(u+1),\ u = 0,1,...,z-1,$$
$$\bar{\mu}_i(z) = \mu_i(z),\ i = 1,2,...,n. \quad (9)$$

Definition 2
A vector

$$\boldsymbol{S}(t,\cdot) = [\boldsymbol{S}(t,0),\boldsymbol{S}(t,1),...,\boldsymbol{S}(t,z)],\ t \in < 0,\infty), \quad (10)$$

where

$$\boldsymbol{S}(t,u) = P(S(t) \ge u \mid S(0) = z) = P(T(u) > t),$$
$$t \in < 0,\infty),\ u = 0,1,...,z, \quad (11)$$

is the probability that the system is in the safety state subset $\{u, u+1,..., z\}$, at the moment t, $t \in < 0,\infty)$, while it was in the safety state z at the moment $t = 0$, is called the safety function of this multistate system.

The safety functions $\boldsymbol{S}(t,u)$, $t \in < 0,\infty)$, $u = 0,1,...,z$, defined by (11) are called the coordinates of the multistate system safety function $\boldsymbol{S}(t,\cdot)$ given by (10). Consequently, the relationship between the distribution function $\boldsymbol{F}(t,u)$ of the system S lifetime $T(u)$ in the safety state subset $\{u, u+1,..., z\}$, and the coordinate $\boldsymbol{S}(t,u)$ of its safety function is given by

$$\boldsymbol{F}(t,u) = P(T(u) \le t) = 1 - P(T(u) > t) = 1 - \boldsymbol{S}(t,u),$$
$$t \in < 0,\infty),\ u = 0,1,...,z.$$

The exemplary graph of a five-state (z = 4) system safety function

$$\boldsymbol{S}(t,\cdot) = [1, \boldsymbol{S}(t,1), \boldsymbol{S}(t,2), \boldsymbol{S}(t,3)],\ t \in < 0,\infty),$$

is shown in Figure 2.

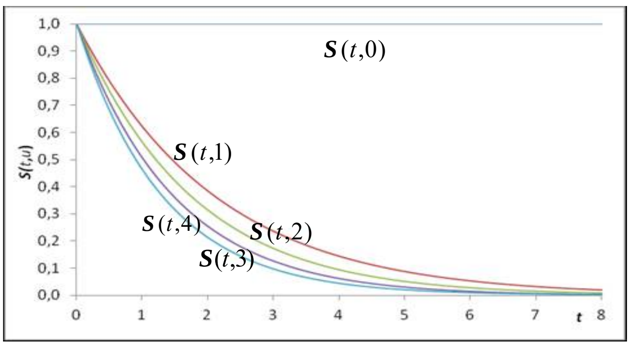

Figure 2. The graph of a five-state system safety function $\boldsymbol{S}(t,\cdot)$ coordinates

Under *Definition 2*, we have

$$\boldsymbol{S}(t,0) \ge \boldsymbol{S}(t,1) \ge \ldots \ge \boldsymbol{S}(t,z),\ t \in < 0,\infty),$$

and if

$$p(t,u) = P(S(t) = u \mid S(0) = z),$$
$$t \in <0,\infty), \; u = 0,1,...,z, \tag{12}$$

is the probability that the system is in the safety state u at the moment t, $t \in <0,\infty)$, while it was in the safety state z at the moment $t = 0$, then

$$\boldsymbol{S}(t,0) = 1, \; \boldsymbol{S}(t,z) = p(t,z), \; t \in <0,\infty), \tag{13}$$

and

$$p(t,u) = \boldsymbol{S}(t,u) - \boldsymbol{S}(t,u+1),$$
$$u = 0,1,...,z-1, \; t \in <0,\infty). \tag{14}$$

Moreover, if

$$\boldsymbol{S}(t,u) = 1 \text{ for } t \le 0, \; u = 1,2,...,z,$$

then

$$\mu(u) = \int_0^\infty \boldsymbol{S}(t,u)dt, \; u = 1,2,...,z, \tag{15}$$

is the mean lifetime of the system in the safety state subset $\{u, u+1,..., z\}$,

$$\sigma(u) = \sqrt{n(u) - [\mu(u)]^2}, \; u = 1,2,...,z, \tag{16}$$

where

$$n(u) = 2\int_0^\infty t\, \boldsymbol{S}(t,u)dt, \; u = 1,2,...,z, \tag{17}$$

is the standard deviation of the system lifetime in the safety state subset $\{u, u+1,..., z\}$, and moreover

$$\bar{\mu}(u) = \int_0^\infty p(t,u)dt, \; u = 1,2,...,z, \tag{18}$$

is the mean lifetime of the system in the safety state u while the integrals (15), (17) and (18) are convergent.
Additionally, according to (13), (14), (15) and (18), we get the following relationship

$$\bar{\mu}(u) = \mu(u) - \mu(u+1), \; u = 0,1,...,z-1,$$
$$\bar{\mu}(z) = \mu(z). \tag{19}$$

Definition 3
A probability

$$\boldsymbol{r}(t) = P(S(t) < r \mid S(0) = z) = P(T(r) \le t),$$
$$t \in <0,\infty),$$

that the system is in the subset of safety states worse than the critical safety state r, $r \in \{1,...,z\}$ while it was in the safety state z at the moment $t = 0$ is called a risk function of the multi-state system [Kołowrocki 2004, 2014], [Kołowrocki, Soszyńska-Budny, 2011].

Under this definition, from (2), we have

$$\boldsymbol{r}(t) = 1 - P(S(t) \ge r \mid S(0) = z) = 1 - \boldsymbol{S}(t,r),$$
$$t \in <0,\infty), \tag{20}$$

and if τ is the moment when the system risk exceeds a permitted level δ, then

$$\tau = \boldsymbol{r}^{-1}(\delta), \tag{21}$$

where $\boldsymbol{r}^{-1}(t)$, if it exists, is the inverse function of the system risk function $\boldsymbol{r}(t)$.
The exemplary graph of a five-state system risk function for the critical safety state $r = 2$

$$\boldsymbol{r}(t) = 1 - \boldsymbol{S}(t,2), \; t \in <0,\infty),$$

corresponding to the safety function illustrated in Figure 2 is shown in Figure 3.

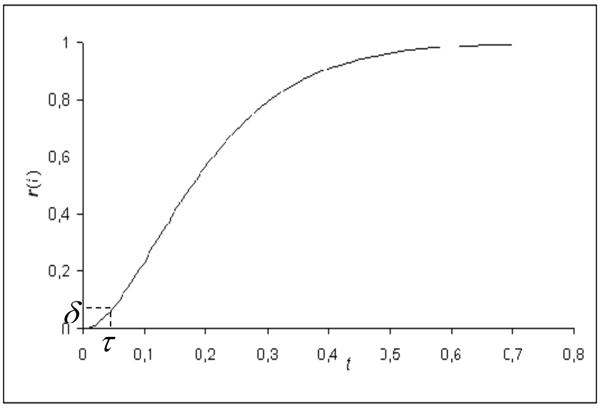

Figure 3. The graph of a five-state system risk function $\boldsymbol{r}(t)$ (the fragility curve)

Now, after introducing the notion of the multistate safety analysis, we may define basic multi-state safety structures.

Definition 4
A multistate system is called series if its lifetime $T(u)$ in the safety state subset $\{u, u+1,..., z\}$, is given by

$$T(u) = \min_{1\le i\le n}\{T_i(u)\},\ u = 1,2,...,z.$$

The number n is called the system structure shape parameter.
The above definition means that a multi-state series system is in the safety state subset $\{u, u+1,..., z\}$, if and only if all its n components are in this subset of safety states. That meaning is very close to the definition of a two-state series system considered in a classical reliability analysis that is not failed if all its components are not failed. This fact can justify the safety structure scheme for a multistate series system presented in Figure 4.

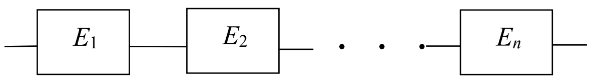

Figure 4.The scheme of a series system safety structure

It is easy to work out that the safety function of the multi-state series system is given by the vector [Kołowrocki 2004, 2014], [Kołowrocki, Soszyńska-Budny, 2011]

$$\boldsymbol{S}(t,\cdot) = [1,\ \boldsymbol{S}(t,1),...,\ \boldsymbol{S}(t,z)] \tag{22}$$

with the coordinates

$$\boldsymbol{S}(t,u) = \prod_{i=1}^{n} S_i(t,u),\ t \in <0,\infty),\ u = 1,2,...,z \tag{23}$$

Definition 5
A multistate system is called parallel if its lifetime $T(u)$ in the safety state subset $\{u,u+1,...,z\}$ is given by

$$T(u) = \max_{1\le i\le n}\{T_i(u)\},\ u = 1,2,...,z.$$

The number n is called the system structure shape parameter.
The above definition means that the multistate parallel system is in the safety state subset $\{u,u+1,...,z\}$ if and only if at least one of its n components is in this subset of safety states. That meaning is very close to the definition of a two-state parallel system in a classical reliability analysis that is not failed if at least one of its components is not failed what can justify the safety structure scheme for a multistate parallel system presented in Figure 5.

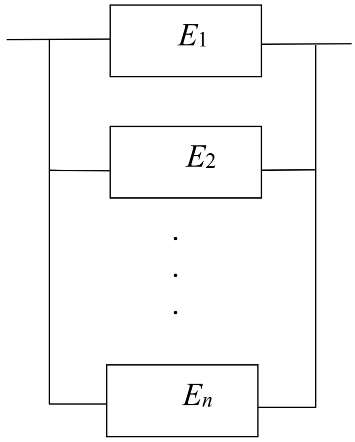

Figure 5. The scheme of a parallel system safety structure

The safety function of the multi-state parallel system is given by the vector [Kołowrocki 2004, 2014], [Kołowrocki, Soszyńska-Budny, 2011]

$$\boldsymbol{S}(t,\cdot) = [1,\boldsymbol{S}(t,1),...,\boldsymbol{S}(t,z)], \tag{24}$$

with the coordinates

$$\boldsymbol{S}(t,u) = 1 - \prod_{i=1}^{n} F_i(t,u),\ t \in <0,\infty),$$

$$u = 1,2,...,z. \tag{25}$$

Definition 6
A multistate system is called an "m out of n" system if its lifetime $T(u)$ in the safety state subset $\{u,u+1,...,z\}$ is given by

$$T(u) = T_{(n-m+1)}(u),\ m = 1,2,...,n,\ u = 1,2,...,z,$$

where $T_{(n-m+1)}(u)$ is the mth maximal order statistic in the sequence of the component lifetimes $T_1(u)$, $T_2(u)$,..., $T_n(u)$, $u = 1,2,...,z$.
The above definition means that the multistate „m out of n" system is in the safety state subset $\{u,u+1,...,z\}$ if and only if at least m out of its n components are in this safety state subset and it is a multistate parallel system if $m = 1$ and it is a multistate series system if $m = n$. The numbers m and n are called the system structure shape parameters. The scheme of an "m out of n" multistate system safety structure, justified in an analogous way as in the case of a multistate series system and a multistate parallel system, is given in Figure 6, where i_1, i_2, …, $i_n \in \{1,2,...,n\}$ and $i_a \ne i_b$ for $a \ne b$.

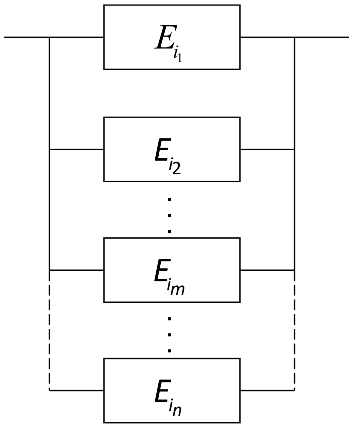

Figure 6. The scheme of an "*m* out of *n*" system safety structure

It can be simply shown that the safety function of the multistate "*m* out of *n*" system is given either by the vector [Kołowrocki 2004, 2014], [Kołowrocki, Soszyńska-Budny, 2011]

$$\boldsymbol{S}(t,\cdot) = [1,\ \boldsymbol{S}(t,1),\ \boldsymbol{S}(t,2),\ ...,\ \boldsymbol{S}(t,z)\,], \tag{26}$$

with the coordinates

$$\boldsymbol{S}(t,u) = 1 - \sum_{\substack{r_1,r_2,...,r_n=0 \\ r_1+r_2+...+r_n \le m-1}}^{1} [S_i(t,u)]^{r_i}\,[F_i(t,u)]^{1-r_i}\ ,$$
$$t \in <0,\infty),\ u = 1,2,...,z, \tag{27}$$

or by the vector

$$\boldsymbol{S}(t,\cdot) = [1,\ \boldsymbol{S}(t,1),\ \boldsymbol{S}(t,2), \ldots, \boldsymbol{S}(t,z)\,], \tag{28}$$

with the coordinates

$$\boldsymbol{S}(t,u) = \sum_{\substack{r_1,r_2,...,r_n=0 \\ r_1+r_2+...+r_n \le \overline{m}}}^{1} [F_i(t,u)]^{r_i}\,[S_i(t,u)]^{1-r_i}\ ,$$
$$t \in <0,\infty),\ \overline{m} = n - m,\ u = 1,2,...,z. \tag{29}$$

Definition 7
A multistate system is called a consecutive "*m* out of *n*: F" system if it is out of the safety state subset $\{u, u+1,..., z\}$ if and only if at least its *m* neighbouring components out of *n* its components arranged in a sequence of $E_1,\ E_2,\ ...,\ E_n$, are out of this safety state subset. The numbers *m* and *n* are called the system structure shape parameters.
After denoting by

$$\boldsymbol{CS}(t,u)\ = P(S(t) \ge u \mid S(0) = z)\ = P(T(u) > t),$$
$$t \in <0,\infty),\ u = 0,1,...,z, \tag{30}$$

the probability that the consecutive "*m* out of *n*: F" system is in the safety state subset $\{u, u+1,..., z\}$ at the moment *t*, $t \in <0,\infty)$, while it was in the safety state *z* at the moment *t* = 0 and by

$$\boldsymbol{CF}(t,u) = 1 - \boldsymbol{CS}(t,u)\ = P(T(u) \le t),$$
$$t \in <0,\infty),\ u = 0,1,...,z, \tag{31}$$

the distribution function of the lifetime $T(u)$ of this system in the safety state subset $\{u,u+1,...,z\}$, while it was in the safety state *z* at the moment *t* = 0, we conclude that the safety function of the consecutive "*m* out of *n*: F" system is the given by the vector

$$\boldsymbol{CS}(t,\cdot) = [1,\ \boldsymbol{CS}(t,1),\ \boldsymbol{CS}(t,2),\ ...,\ \boldsymbol{CS}(t,z)\,], \tag{32}$$

with the coordinates given by the following recurrent formula [Guze, Kolowrocki, 2008], [Kołowrocki 2004, 2014], [Kołowrocki, Soszyńska-Budny, 2011]

$$\boldsymbol{CS}(t,u)$$
$$= \begin{cases} 1 & \text{for } n < m, \\ 1 - \prod_{i=1}^{n} F_i(t,u) & \text{for } n = m, \\ S_n(t,u)\boldsymbol{CS}_{n-1}(t,u) & \\ + \sum_{i=1}^{m-1} S_{n-i}(t,u)\boldsymbol{CS}_{n-i-1}(t,u) & \\ \cdot \prod_{j=n-i+1}^{n} F_j(t,u) & \text{for } n > m, \end{cases} \tag{33}$$

for $t \ge 0$, $u = 1,2,...,z$.

Other basic multistate safety structures with components degrading in time series-parallel, parallel-series, series-"*m* out of *k*", "m_i out of l_i"-series, series-consecutive "*m* out of *k*: F" and consecutive "m_i out of l_i: F"-series systems. To define them, we assume that:

- *k* is the number of the system subsystems,
- l_i, $i = 1,2,...,k$, are the numbers of the subsystem components,
- E_{ij}, $i = 1,2,...,k$, $j = 1,2,...,l_i$, k, l_1, l_2,..., $l_k \in N$, are components of a system,
- all components E_{ij} have the same safety state set as before $\{0,1,...,z\}$,
- $T_{ij}(u)$, $i = 1,2,...,k$, $j = 1,2,...,l_i$, k, l_1, l_2,..., $l_k \in N$, are independent random variables representing the lifetimes of components E_{ij} in the safety state subset $\{u, u+1,..., z\}$ while they were in the safety state *z* at the moment *t* = 0,

– $E_{ij}(t)$ is a component E_{ij} safety state at the moment t, $t \in <0,\infty)$, while they were in the safety state z at the moment $t = 0$.

Definition 8
A vector

$$S_{ij}(t,\cdot) = [S_{ij}(t,0), S_{ij}(t,1), ..., S_{ij}(t,z)],\ t \in <0,\infty),$$
$$i = 1,2,...,k,\ j = 1,2,...,l_i, \tag{34}$$

where

$$S_{ij}(t,u) = P(E_{ij}(t) \geq u \mid E_{ij}(0) = z) = P(T_{ij}(u) > t),$$
$$t \in <0,\infty),\ u = 0,1,...,z, \tag{35}$$

is the probability that the component E_{ij} is in the safety state subset $\{u, u+1, ..., z\}$ at the moment t, $t \in <0,\infty)$, while it was in the safety state z at the moment $t = 0$, is called the safety function of a multistate component E_{ij}.

The safety functions $S_{ij}(t,u)$, $t \in <0,\infty)$, $u = 0,1,...,z$, defined by (35) are called the coordinates of the component E_{ij}, $i = 1,2,...,k$, $j = 1,2,...,l_i$, safety function $R_{ij}(t,\cdot)$ given by (34). Thus, the relationship between the distribution function $F_{ij}(t,u)$ of the component E_{ij}, $i = 1,2,...,k$, $j = 1,2,...,l_i$, lifetime $T_{ij}(u)$ in the safety state subset $\{u, u+1, ..., z\}$ and the coordinate $S_{ij}(t,u)$ of its safety function is given by

$$F_{ij}(t,u) = P(T_{ij}(u) \leq t) = 1 - P(T_{ij}(u) > t)$$
$$= 1 - S_{ij}(t,u),\ t \in <0,\infty),\ u = 0,1,...,z.$$

Definition 9
A multistate system is called series-parallel if its lifetime $T(u)$ in the state subset $\{u, u+1, ..., z\}$ is given by

$$T(u) = \max_{1 \leq i \leq k}\{\min_{1 \leq j \leq l_i}\{T_{ij}(u)\}\},\ u = 1,2,...,z.$$

The above definition means that the multistate series-parallel system is composed of k multistate series subsystems and it is in the safety state subset $\{u, u+1, ..., z\}$ if and only if at least one out of its k series subsystems is in this safety state subset. In this definition, l_i, $i = 1,2,...,k$, denote the numbers of components in the series subsystems. The numbers k and l_1, l_2,..., l_k are called the system structure shape parameters. Joining the justification for the safety structure schemes of the multistate series system and the multistate parallel system leads to the scheme of a multistate series-parallel system safety structure given in Figure 7.

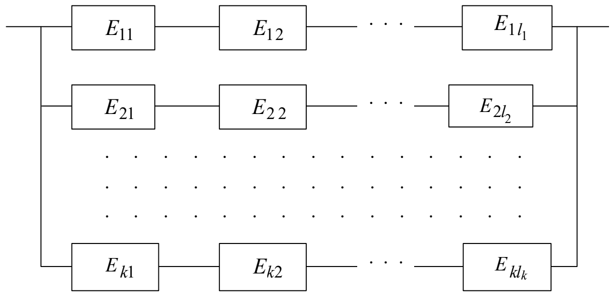

Figure 7. The scheme of a series-parallel system safety structure

The safety function of the multi-state series-parallel system is given by the vector [Kołowrocki 2004, 2014], [Kołowrocki, Soszyńska-Budny, 2011]

$$\boldsymbol{S}(t,\cdot) = [1,\ \boldsymbol{S}(t,1),\ \boldsymbol{S}(t,2),\ ...,\ \boldsymbol{S}(t,z)], \tag{36}$$

with the coordinates

$$\boldsymbol{S}(t,u) = 1 - \prod_{i=1}^{k}[1 - \prod_{j=1}^{l_i} S_{ij}(t,u)],$$
$$t \in <0,\infty),\ u = 1,2,...,z, \tag{37}$$

where k is the number of series subsystems linked in parallel and l_i are the numbers of components in the series subsystems.

Definition 10
A multistate system is called parallel-series if its lifetime $T(u)$ in the safety state subset $\{u, u+1, ..., z\}$ is given by

$$T(u) = \min_{1 \leq i \leq k}\{\max_{1 \leq j \leq l_i}\{T_{ij}(u)\}\},\ u = 1,2,...,z.$$

The above definition means that the multistate parallel-series system is composed of k multistate parallel subsystems and it is in the safety state subset

$\{u, u+1, ..., z\}$ if and only if all its k parallel subsystems are in this safety state subset. In this definition l_i, $i = 1,2,...,k$, denote the numbers of components in the parallel subsystems. The numbers k and $l_1, l_2, ..., l_k$ are called the system structure shape parameters. Joining the justification for the safety structure schemes of the multistate parallel system and the multistate series system leads to the scheme of a multistate parallel-series system safety structure given in Figure 8.

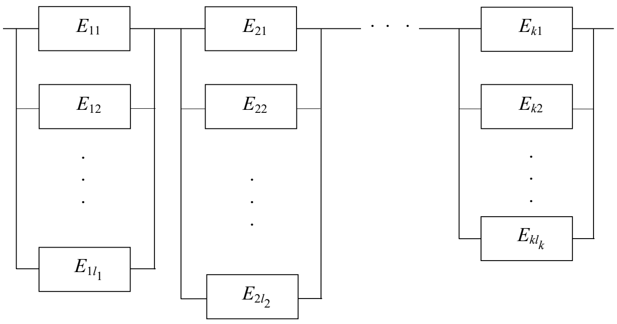

Figure 8. The scheme of a parallel-series system safety structure

The safety function of the multi-state parallel-series system is given by the vector [Kołowrocki 2004, 2014], [Kołowrocki, Soszyńska-Budny, 2011]

$$\boldsymbol{S}(t,\cdot) = [1,\ \boldsymbol{S}(t,1),\ \boldsymbol{S}(t,2),\ ...,\ \boldsymbol{S}(t,z)\,], \tag{38}$$

with the coordinates

$$\boldsymbol{S}(t,u) = \prod_{i=1}^{k}[1 - \prod_{j=1}^{l_i} F_{ij}(t,u)],\ t \in <0,\infty),$$
$$u = 1,2,...,z, \tag{39}$$

where k is the number of its parallel subsystems linked in series and l_i, $i = 1,2,...,k$, are the numbers of components in the parallel subsystems.

Definition 11
A multistate system is called a series-"m out of k" system if its lifetime $T(u)$ in the safety state subset $\{u, u+1, ..., z\}$ is given by

$$T(u) = T_{(k-m+1)}(u),\ m = 1,2,...,k,\ u = 1,2,...,z,$$

where $T_{(k-m+1)}(u)$ is the mth maximal order statistic in the set of random variables

$$T_i(u) = \min_{1\le j\le l_i}\{T_{ij}(u)\},\ i = 1,2,...,k,\ u = 1,2,...,z.$$

The above definition means that the multistate series-"m out of k" system is composed of k multi-state series subsystems and it is in the safety state subset $\{u, u+1, ..., z\}$ if and only if at least m out of its k series subsystems are in this safety state subset. In this definition l_i, $i = 1,2,...,k$, denote the numbers of components in the series subsystems. The numbers m, k and $l_1, l_2, ..., l_k$ are called the system structure shape parameters. Joining the justification for the safety structure schemes of the multistate series system and the multistate "m out of k" system leads to the scheme of a multistate series-"m out of k" system safety structure given in Figure 9, where $i_1, i_2, ... i_k \in \{1,2,...,k\}$ and $i_a \ne i_b$ for $a \ne b$.

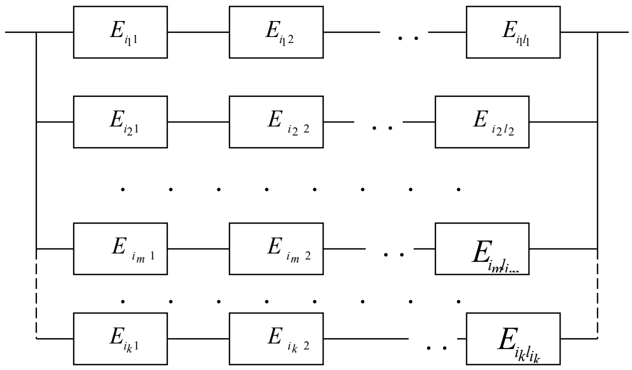

Figure 9. The scheme of a series-"m out of k" system safety structure

The safety function of the multi-state series-"m out of k" system is given either by the vector [Kołowrocki 2004, 2014], [Kołowrocki, Soszyńska-Budny, 2011]

$$\boldsymbol{S}(t,\cdot) = [1,\ \boldsymbol{S}(t,1),\ \boldsymbol{S}(t,2),\ ...,\ \boldsymbol{S}(t,z)\,], \tag{40}$$

with the coordinates

$$\boldsymbol{S}(t,u)$$
$$= 1 - \sum_{\substack{r_1,r_2,...,r_k=0 \\ r_1+r_2+...+r_k \le m-1}}^{1} \prod_{i=1}^{k}[[\prod_{j=1}^{l_i} S_{ij}(t,u)]^{r_i}[1 - \prod_{j=1}^{l_i} S_{ij}(t,u)]^{1-r_i},$$
$$t \in <0,\infty),\ u = 1,2,...,z, \tag{41}$$

or by the vector

$$\boldsymbol{S}(t,\cdot) = [1,\ \boldsymbol{S}(t,1),\ \boldsymbol{S}(t,2),\ ...,\ \boldsymbol{S}(t,z)\,], \tag{42}$$

with the coordinates

$$\boldsymbol{S}(t,u)$$

$$= \sum_{\substack{r_1,r_2,...,r_k=0 \\ r_1+r_2+...+r_k \le \overline{m}}}^{1} \prod_{i=1}^{k} [1 - \prod_{j=1}^{l_i} S_{ij}(t,u)]^{r_i} [\prod_{j=1}^{l_i} S_{ij}(t,u)]^{1-r_i},$$

$$t \in <0,\infty), \ \overline{m} = k - m, \ u = 1,2,...,z. \qquad (43)$$

Definition 12
A multistate system is called an "m_i out of l_i"-series system if its lifetime $T(u)$ in the safety state subset $\{u, u+1, ..., z\}$ is given by

$$T(u) = \min_{1 \le i \le k} T_{(l_i - m_i + 1)}(u), \ m_i = 1,2,..., l_i, \ u = 1,2,...,z,$$

Where $T_{(l_i - m_i + 1)}(u)$ is the m_ith maximal order statistic in the set of random variables

$$T_{i1}(u), \ T_{i2}(u), \ ..., \ T_{il_i}(u), \ i = 1,2,...,k, \ u = 1,2,...,z.$$

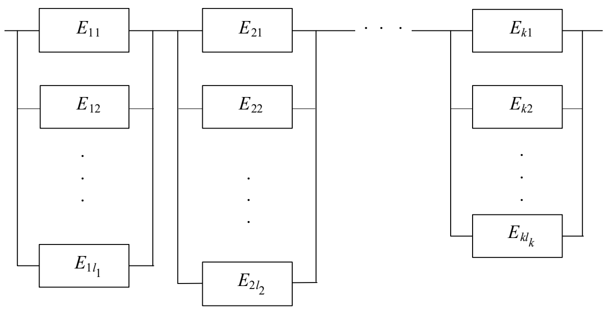

Figure 10. The scheme of an "m_i out of l_i"-series system safety structure

The above definition means that the multi-state "m_i out of l_i"-series system is composed of k subsystems that are multi-state "m_i out of l_i" systems and it is in the safety state subset $\{u, u+1, ..., z\}$ if all its "m_i out of l_i" subsystems are in this safety state subset. In this definition l_i, $i = 1,2,...,k$, denote the numbers of components in the "m_i out of l_i" subsystems. The numbers k, m_1, m_2, ..., m_k and l_1, l_2,..., l_k are called the system structure shape parameters. Joining the justification for the safety structure schemes of the multistate "m_i out of l_i" system and the multistate series system schemes leads to the scheme of a multistate of an "m_i out of l_i"-series system safety structure given in Figure 10, where $j_1, j_2, ..., j_{l_i} \in \{1,2,...,l_i\}$, for $i = 1,2,...,k$ and $i_a \ne i_b$ for $a \ne b$.
The safety function of the multi-state "m_i out of l_i"-series system is given either by the vector [Kołowrocki 2004, 2014], [Kołowrocki, Soszyńska-Budny, 2011]

$$\boldsymbol{S}(t,\cdot) = [1, \ \boldsymbol{S}(t,1), \ \boldsymbol{S}(t,2), \ ..., \ \boldsymbol{S}(t,z)], \qquad (44)$$

where

$$\boldsymbol{S}(t,u) = \prod_{i=1}^{k} [\sum_{\substack{r_1,r_2,...,r_{l_i}=0 \\ m_i \le r_1+r_2+...+r_{l_i} \le l_i}}^{1} \prod_{j=1}^{l_i} [S_{ij}(t,u)]^{r_j} [1 - S_{ij}(t,u)]^{1-r_j}],$$

$$t \in <0,\infty), \ u = 1,2,...,z, \qquad (45)$$

or by the vector

$$\boldsymbol{S}(t,\cdot) = [1, \ \boldsymbol{S}(t,1), \ \boldsymbol{S}(t,2), \ ..., \ \boldsymbol{S}(t,z)], \qquad (46)$$

where

$$\boldsymbol{S}(t,u) = \prod_{i=1}^{k} [1 - \sum_{\substack{r_1,r_2,...,r_{l_i}=0 \\ r_1+r_2+...+r_{l_i} \le m_i - 1}}^{1} \prod_{j=1}^{l_i} [S_{ij}(t,u)]^{r_j} [1 - S_{ij}(t,u)]^{1-r_j}],,$$

$$t \in <0,\infty), \ i = 1,2, \ldots, k, \ u = 1,2,...,z. \qquad (47)$$

To define a multistate series-consecutive "m out of k: F" system, we assume that components $E_{i1}, E_{i2}, ..., E_{il_i}$,, $i = 1,2,...,k$, create a series subsystems S_i, $i = 1,2,...,k$, and that these subsystems are arranged in a sequence $S_1, S_2, ..., S_k$.

Definition 13
A multi-state system is called series-consecutive "m out of k: F" system if it is out of the safety state subset $\{u, u+1, ..., z\}$ if and only if at least its m neighbouring series subsystems out of k its series subsystems arranged in a sequence $S_1, S_2, ..., S_k$, are out of this safety state subset. The numbers m, k and l_1, l_2,..., l_k are called the system structure shape parameters.
According to the above definition and formulae (22)-(23) the coordinates of the safety function of the subsystem S_i, $i = 1,2,...,k$, are given by

$$\boldsymbol{S}_{il_i}(t,u) = \prod_{j=1}^{l_i} S_{ij}(t,u), \ t \in <0,\infty), \ u = 1,2,...,z,$$

and its lifetime distribution functions are given by

$$\boldsymbol{F}_{il_i}(t,u) = 1 - \boldsymbol{S}_{il_i}(t,u) = 1 - \prod_{j=1}^{l_i} S_{ij}(t,u), \ t \in <0,\infty), \ u = 1,2,...,z, \ i = 1,2,..., k.$$

Hence, considering (32)-(33) , the safety function of the multi-state series-consecutive "*m* out of *k*: F" system is given by the vector

$$\boldsymbol{CS}(t,\cdot) = [1,\ \boldsymbol{CS}(t,1),\ \boldsymbol{CS}(t,2),\ ...,\ \boldsymbol{CS}(t,z)], \tag{48}$$

with the coordinates given by the following recurrent formula

$$\boldsymbol{CS}(t,u)$$
$$= \begin{cases} 1 & \text{for } k < m, \\ 1 - \prod_{i=1}^{k}[1 - \prod_{j=1}^{l_i} S_{ij}(t,u)] & \text{for } k = m, \\ [\prod_{j=1}^{l_k} S_{kj}(t,u)]\boldsymbol{CS}_{k-1;l_1,l_2,...,l_k}(t,u) & \\ + \sum_{j=1}^{m-1}[\prod_{v=1}^{l_{k-j}} S_{k-j,v}(t,u)]\boldsymbol{CS}_{k-j-1,l_1,l_2,...,l_k}(t,u) & \\ \cdot \prod_{i=k-j+1}^{k}[1 - \prod_{v=1}^{l_i} S_{iv}(t,u)] & \text{for } k > m, \end{cases}$$

for $t \geq 0$, $u = 1,2,...,z$. (49)

To define a multistate consecutive "*m* out of *l*: F"-series system, we assume that components $E_{i1}, E_{i2}, ..., E_{il_i}$, $i = 1,2,\ldots,k$, create a consecutive "m_i out of l_i: F" subsystem S_i, $i = 1,2,\ldots,k$, and that these subsystems $S_1, S_2, \ldots, S_k$, create a series system.

Definition 14
A multi-state system is called a consecutive "m_i out of l_i: F"-series system if it is out of the safety state subset $\{u, u+1,...,z\}$ if and only if at least one of its consecutive "m_i out of l_i: F" subsystems S_i, $i = 1,2,\ldots,k$, is out of this safety state subset. The numbers k and m_i, l_i, $i = 1,2,\ldots,k$, are called the system structure shape parameters.

According to the above definition and formulae (32)-(33), the safety function of the subsystem S_i, $i = 1,2,\ldots,k$, is the vector

$$\boldsymbol{CS}_{i,l_i}(t,\cdot) = [1,\ \boldsymbol{CS}_{i,l_i}(t,1),\ \boldsymbol{CS}_{i,l_i}(t,2),\ ...,\ \boldsymbol{CS}_{i,l_i}(t,z)], \tag{50}$$

with the coordinates given by the following recurrent formulae

$$\boldsymbol{CS}_{i,l_i}(t,u)$$
$$= \begin{cases} 1 & \text{for } l_i < m_i, \\ 1 - \prod_{j=1}^{l_i} F_{ij}(t,u) & \text{for } l_i = m_i, \\ S_{il_i}(t,u)\boldsymbol{CS}_{i,l_i-1}(t,u) + & \\ \sum_{j=1}^{m_i-1} S_{i,l_i-j}(t,u)\boldsymbol{CS}_{i,l_i-j-1}(t,u) & \\ \cdot \prod_{v=l_i-j+1}^{l_i} F_{iv}(t,u) & \text{for } l_i > m_i, \end{cases} \tag{51}$$

for $t \geq 0$, $u = 1,2,...,z$.

Hence, the safety function of the multi-state consecutive "m_i out of l_i: F"-series system , we conclude that it is given by the vector

$$\boldsymbol{CS}(t,\cdot) = [1,\ \boldsymbol{CS}(t,1),\ \boldsymbol{CS}(t,2),\ ...,\ \boldsymbol{CS}(t,z)], \tag{52}$$

with the coordinates

$$\boldsymbol{CS}(t,u) = \prod_{i=1}^{k} \boldsymbol{CS}_{i,l_i}(t,u) \text{ for } t \geq 0,$$
$$u = 1,2,...,z, \tag{53}$$

where $\boldsymbol{CS}_{i,l_i}(t,u)$, i=1, 2,… , k, are given by the recurrent formula (51).
Further, for the systems composed of components having multistate exponential safety functions, we formulate the particular cases of the fixed above formulae for the considered system structures in the form of the following proposition.

Proposition 1
If components of the multi-state system have the exponential safety functions

$$S_i(t,\cdot) = [1, S_i(t,1),\ldots,S_i(t,z)],\ t \in (-\infty,\infty), \tag{54}$$

where

$$S_i(t,u) = 1 \text{ for } t < 0,\ S_i(t,u) = \exp[-\lambda_i(u)t]$$
$$\text{for } t \geq 0,\ \lambda_i(u) > 0,\ i = 1,2,...,n,\ u = 1,2,\ldots,z, \tag{55}$$

in the case of the multistate series, parallel, "m out of n", consecutive "m out of n" systems and respectively

$$S_{ij}(t,\cdot) = [1, S_{ij}(t,1),\ldots,S_{ij}(t,z)],\ t \in (-\infty,\infty), \tag{56}$$

where

$S_{ij}(t,u) = 1$ for $t < 0$, $S_{ij}(t,u) = \exp[-\lambda_{ij}(u)t]$
for $t \geq 0$, $\lambda_{ij}(u) > 0$, $i = 1,2,...,n, j = 1,2,...,l_i$,
$u = 1,2,\ldots,z$, (57)

in the case of the series-parallel, parallel-series, series-"m out of k", "m_i out of l_i"-series, series-consecutive "m out of k" and consecutive "m_i out of l_i"-series systems, then its safety function is given by the vector:

i) for a series system

$$\boldsymbol{S}(t,\cdot) = [1,\ \boldsymbol{S}(t,1), \boldsymbol{S}(t,2),\ ...,\ \boldsymbol{S}(t,z)], \tag{58}$$

where

$\boldsymbol{S}(t,u) = 1$ for $t < 0$, $\boldsymbol{S}(t,u) = \exp[-\sum_{i=1}^{n}\lambda_i(u)t]$
for $t \geq 0$, $u = 1,2,...,z$, (59)

ii) for a parallel system

$$\boldsymbol{S}(t,\cdot) = [1,\ \boldsymbol{S}(t,1), \boldsymbol{S}(t,2),\ ...,\ \boldsymbol{S}(t,z)], \tag{60}$$

where

$\boldsymbol{S}(t,u) = 1$ for $t < 0$,
$\boldsymbol{S}(t,u) = 1 - \prod_{i=1}^{n}[1 - \exp[-\lambda_i(u)t]]$ for $t \geq 0$,
$u = 1,2,...,z$, (61)

iii) for a "m out of n" system

$$\boldsymbol{S}(t,\cdot) = [1,\ \boldsymbol{S}(t,1), \boldsymbol{S}(t,2),\ ...,\ \boldsymbol{S}(t,z)], \tag{62}$$

where

$\boldsymbol{S}(t,u) = 1$ for $t < 0$,

$$\boldsymbol{S}(t,u) = 1 - \sum_{\substack{r_1,r_2,...,r_n=0\\ r_1+r_2+...+r_n \leq m-1}}^{1} \prod_{i=1}^{n} \exp[-r_i\lambda_i(u)t][1 - \exp[-\lambda_i(u)t]]^{1-r_i}$$

for $t \geq 0$, $u = 1,2,...,z$, (63)

or

$$\boldsymbol{S}(t,\cdot) = [1,\ \boldsymbol{S}(t,1), \boldsymbol{S}(t,2),\ ...,\ \boldsymbol{S}(t,z)], \tag{64}$$

where

$\boldsymbol{S}(t,u) = 1$ for $t < 0$,

$$\boldsymbol{S}(t,u) = \sum_{\substack{r_1,r_2,...,r_n=0\\ r_1+r_2+...+r_n \leq \bar{m}}}^{1} \prod_{i=1}^{n}[1 - \exp[-\lambda_i(u)t]]^{r_i} \exp[-(1-r_i)\lambda_i(u)t]$$

for $t \geq 0$, $\bar{m} = n - m$, $u = 1,2,...,z$, (65)

iv) for a consecutive "m out of n: F" system

$$\boldsymbol{CS}(t,\cdot) = [1,\ \boldsymbol{CS}(t,1),\ \boldsymbol{CS}(t,2),\ ...,\ \boldsymbol{CS}(t,z)], \tag{66}$$

where

$\boldsymbol{CS}(t,u) = 1$ for $t < 0$,

$$\boldsymbol{CS}(t,u) = \boldsymbol{CS}_n(t,u)$$

$$= \begin{cases} 1 & \text{for } n < m, \\ 1 - \prod_{i=1}^{n}[1 - \exp[-\lambda_i(u)t]] & \text{for } n = m, \\ \exp[-\lambda_n(u)t]\boldsymbol{CS}_{n-1}(t,u) & \\ \quad + \sum_{i=1}^{m-1}\exp[-\lambda_{n-i}(u)t]\boldsymbol{CS}_{n-i-1}(t,u) & \\ \quad \cdot \prod_{j=n-i+1}^{n}[1 - \exp[-\lambda_j(u)t]] & \text{for } n > m, \end{cases} \tag{67}$$

for $t < 0$, $u = 1,2,...,z$,

v) for a series-parallel system

$$\boldsymbol{S}(t,\cdot) = [1,\ \boldsymbol{S}(t,1), \boldsymbol{S}(t,2),\ ...,\ \boldsymbol{S}(t,z)], \tag{68}$$

where

$\boldsymbol{S}(t,u) = 1$ for $t < 0$,
$\boldsymbol{S}(t,u) = 1 - \prod_{i=1}^{k}[1 - \exp[-\sum_{j=1}^{l_i}\lambda_{ij}(u)t]]$ for $t \geq 0$,
$u = 1,2,\ldots,z$, (69)

vi) for a parallel-series system

$$\boldsymbol{S}(t,\cdot) = [1,\ \boldsymbol{S}(t,1), \boldsymbol{S}(t,2),\ ...,\ \boldsymbol{S}(t,z)], \tag{70}$$

where

$\boldsymbol{S}(t,u) = 1$ for $t < 0$,

$$\boldsymbol{S}(t,u) = \prod_{i=1}^{k}[1-\prod_{j=1}^{l_i}[1-\exp[-\lambda_{ij}(u)t]]] \text{ for } t \geq 0,$$
$$u = 1,2,...,z, \tag{71}$$

vii) for a series-“*m* out of *k*” system

$$\boldsymbol{S}(t,\cdot) = [1, \boldsymbol{S}(t,1), \boldsymbol{S}(t,2), ..., \boldsymbol{S}(t,z)], \tag{72}$$

where

$$\boldsymbol{S}(t,u) = 1 \text{ for } t < 0,$$

$$\boldsymbol{S}(t,u) = 1$$
$$- \sum_{\substack{r_1,r_2,...,r_k=0 \\ r_1+r_2+...+r_k \leq m-1}}^{1} \prod_{i=1}^{k}\exp[-r_i\sum_{j=1}^{l_i}\lambda_{ij}(u)t][1-\exp[-\sum_{j=1}^{l_i}\lambda_{ij}(u)t]]^{1-r_i}$$
$$\text{for } t \geq 0, \ u = 1,2,...,z, \tag{73}$$

or

$$\boldsymbol{S}(t,\cdot) = [1, \boldsymbol{S}(t,1), \boldsymbol{S}(t,2), ..., \boldsymbol{S}(t,z)], \tag{74}$$

where

$$\boldsymbol{S}(t,u) = 1 \text{ for } t < 0,$$

$$\boldsymbol{S}(t,u)$$
$$= \sum_{\substack{r_1,r_2,...,r_k=0 \\ r_1+r_2+...+r_k \leq \overline{m}}}^{1} \prod_{i=1}^{k}[1-\exp[-\sum_{j=1}^{l_i}\lambda_{ij}(u)t]]^{r_i}\exp[-(1-r_i)\sum_{j=1}^{l_i}\lambda_{ij}(u)t]$$
$$\text{for } \quad t \geq 0, \ \overline{m} = k-m, \ u = 1,2,\ldots,z, \tag{75}$$

viii) for a “m_i out of l_i”-series system

$$\boldsymbol{S}(t,\cdot) = [1, \boldsymbol{S}(t,1), \boldsymbol{S}(t,2), ..., \boldsymbol{S}(t,z)], \tag{76}$$

where

$$\boldsymbol{S}(t,u) = 1 \text{ for } t < 0,$$

$$\boldsymbol{S}(t,u) = \prod_{i=1}^{k}[1 - \sum_{\substack{r_1,r_2,...,r_{l_i}=0 \\ r_1+r_2+...+r_{l_i} \leq m_i-1}}^{1} \exp[-r_i\sum_{j=1}^{l_i}\lambda_{ij}(u)t]$$
$$\prod_{j=1}^{li}[1-\exp[-\lambda_{ij}(u)t]]^{1-r_i}] \text{ for } t \geq 0,$$
$$u = 1,2,\ldots,z, \tag{77}$$

or

$$\boldsymbol{S}(t,\cdot) = [1, \boldsymbol{S}(t,1), \boldsymbol{S}(t,2), ..., \boldsymbol{S}(t,z)], \tag{78}$$

where

$$\boldsymbol{S}(t,u) = 1 \text{ for } t < 0,$$

$$\boldsymbol{S}(t,u)$$
$$= \prod_{i=1}^{k}[\sum_{\substack{r_1,r_2,...,r_{l_i}=0 \\ r_1+r_2+...+r_{l_i} \leq \overline{m}_i}}^{1} \prod_{j=1}^{li}[1-\exp[-\lambda_{ij}(u)t]]^{r_i}\exp[-(1-r_i)\sum_{j=1}^{l_i}\lambda_{ij}(u)t]]$$
$$\text{for } t \geq 0, \ \overline{m}_i = l_i - m_i, \tag{79}$$

$i = 1,2, \ldots, k$, $u = 1,2,\ldots,z$.

ix) for a series-consecutive “*m* out of *k*: F” system

$$\boldsymbol{CS}(t,\cdot) = [1, \boldsymbol{CS}(t,1), \boldsymbol{CS}(t,2), ..., \boldsymbol{CS}(t,z)], \tag{80}$$

where

$$\boldsymbol{CS}(t,u) = 1 \text{ for } t < 0,$$

$$\boldsymbol{CS}(t,u) = \boldsymbol{CS}_k(t,u)$$
$$= \begin{cases} 1 & \text{for } k < m, \\ 1-\prod_{i=1}^{k}[1-\exp[-\sum_{j=1}^{l_i}\lambda_{ij}(u)t]] & \text{for } k = m, \\ \exp[-\sum_{j=1}^{l_k}\lambda_{kj}(u)t]\boldsymbol{CS}_{k-1;l_1,l_2,...l_k}(t,u) & \\ +\sum_{j=1}^{m-1}[\exp[-\sum_{v=1}^{l_{k-j}}\lambda_{k-j,v}(u)t]] & \\ \boldsymbol{CS}_{k-j-1;l_1,l_2,...,l_k}(t,u) & \\ \cdot\prod_{i=k-j+1}^{k}[1-\exp[-\sum_{v=1}^{l_i}\lambda_{iv}(u)t]] & \text{for } k > m, \end{cases} \tag{81}$$

$$\text{for } t \geq 0, \ u = 1,2,\ldots,z.$$

x) for a consecutive “m_i out of l_i: F”-series system

$$\boldsymbol{CS}(t,\cdot) = [1, \boldsymbol{CS}(t,1), \boldsymbol{CS}(t,2), ..., \boldsymbol{CS}(t,z)], \tag{82}$$

where

$$\boldsymbol{CS}(t,u) = 1 \text{ for } t < 0,$$
$$\boldsymbol{CS}(t,u) = \prod_{i=1}^{k}\boldsymbol{CS}_{i,l_i}(t,u) \quad \text{for } t \geq 0,$$
$$u = 1,2,\ldots,z, \tag{83}$$

where $\boldsymbol{CS}_{i,l_i}(t,u)$, i=1, 2,… , k, are given by

$$\boldsymbol{CS}_{i,l_i}(t,u)$$
$$= \begin{cases} 1 & \text{for } l_i < m_i, \\ 1 - \prod\limits_{j=1}^{l_i} [1 - \exp[-\lambda_{ij}(u)t)] & \text{for } l_i = m_i, \\ \exp[-\lambda_{il_i}(u)t]\boldsymbol{CS}_{i,l_i-1}(t,u) & \\ + \sum\limits_{j=1}^{m_i-1} \exp[-\lambda_{il_i-j}(u)t]\boldsymbol{CS}_{l_i\text{-}j\text{-}1}(t,u) & \\ \cdot \prod\limits_{v=l_i-j+1}^{l_i} [1 - \exp[-\lambda_{iv}(u)t]]] & \text{for } l_i > m_i, \end{cases} \qquad (84)$$

for $t \ge 0$, $u = 1,2,\ldots,z$.

We complete our considerations giving a practically very important tool concerned with the safety function of the multistate series system composed of the subsystems which are the multistate systems earlier considered in the report.

Proposition 2
The safety function of the series system composed of n multistate subsystems S_i, $i = 1,2,\ldots,n$, each of them is one of the homogeneous or non-homogeneous series, parallel, "*m* out of *n*", consecutive "*m* out of *n*: F", series-parallel, parallel-series, series-"*m* out of *n*", "*m* out of *n*"-series, series-consecutive "*m* out of *n*: F" and consecutive "*m* out of *n*: F"-series multistate system is given by the vector

$$\boldsymbol{S}(t,\cdot) = [1,\ \boldsymbol{S}(t,1),\ \boldsymbol{S}(t,2),\ \ldots,\ \boldsymbol{S}(t,z)\,], \qquad (85)$$

with the coordinates

$$\boldsymbol{S}(t,u) = \prod_{i=1}^{n} \boldsymbol{S}_i(t,u),\ t \in \langle 0,\infty),\ u = 1,2,\ldots,z, \qquad (86)$$

where $\boldsymbol{S}_i(t,u)$, $i = 1,2,\ldots,n$, are the coordinates of the safety functions of the subsystems S_i, $i = 1,2,\ldots,n$, determined by the appropriate formulae dependently of the subsystem kind.

3. Conclusions

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