Journal of Polish Safety and Reliability Association
Summer Safety and Reliability Seminars, Volume 7, Number 3, 2016

Kołowrocki Krzysztof

Soszyńska-Budny Joanna
Maritime University, Gdynia, Poland

Safety of multistate ageing systems

Keywords

safety function, risk function, multistate system, ageing, oil pipeline system

Abstract

Basic notions of the ageing multistate systems safety analysis are introduced. The system components and the system multistate safety functions are defined. The mean values and variances of the multistate systems lifetimes in the safety state subsets and the mean values of their lifetimes in the particular safety states are defined. The multi-state system risk function and the moment of exceeding by the system the critical safety state are introduced. The exemplary safety structures of the multistate systems with ageing components are defined and their safety functions are determined. As a particular case, the safety functions of the considered multistate systems composed of components having exponential safety functions are determined. Applications of the proposed multistate system safety models to the evaluation and prediction of the safty characteristics of the oil piping transportation system is presented as well.

1. Introduction

Taking into account the importance of the safety and operating process effectiveness of real technical systems it seems reasonable to expand the two-state approach [11], [14] to multi-state approach [1]-[4], [9]-[10], [14], [18]-[20] in safety analysis. The assumption that the systems are composed of multi-state components with safety states degrading in time [11], [14], [18]-[20] gives the possibility for more precise analysis of their safety and operational processes' effectiveness. This assumption allows us to distinguish a system safety critical state to exceed which is either dangerous for the environment or does not assure the necessary level of its operation process effectiveness. Then, an important system safety characteristic is the time to the moment of exceeding the system safety critical state and its distribution, which is called the system risk function. This distribution is strictly related to the system safety function that are basic characteristics of the multi-state system. The safety models of the considered here typical multistate system structures can be applied in the safety analysis of real complex technical systems. They may be successfully applied, for instance, to safety analysis, identification, prediction and optimization of the critical infrastructures.

2. Safety analysis of multistate systems

In the multistate safety analysis to define the system with degrading components, we assume that:

- n is the number of the system components,
- E_i, $i = 1,2,...,n$, are components of a system,
- all components and a system under consideration have the safety state set $\{0,1,...,z\}$, $z \geq 1$,
- the safety states are ordered, the safety state 0 is the worst and the safety state z is the best,
- $T_i(u)$, $i = 1,2,...,n$, are independent random variables representing the lifetimes of components E_i in the safety state subset $\{u,u+1,...,z\}$, while they were in the safety state z at the moment $t = 0$,
- $T(u)$ is a random variable representing the lifetime of a system in the safety state subset $\{u,u+1,...,z\}$ while it was in the safety state z at the moment $t = 0$,
- the system states degrades with time t,
- $s_i(t)$ is a component E_i safety state at the moment t, $t \in <0,\infty)$, given that it was in the safety state z at the moment $t = 0$,
- $s(t)$ is a system S safety state at the moment t, $t \in <0,\infty)$, given that it was in the safety state z at the moment $t = 0$.

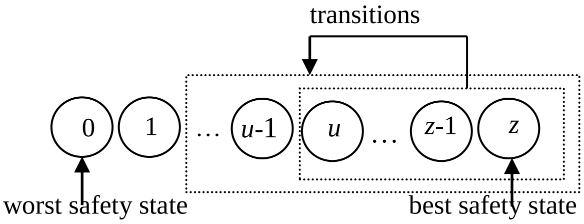

Figure 1. Illustration of a system and components safety states changing

The above assumptions mean that the safety states of the system with degrading components may be changed in time only from better to worse [6], [11], [14], [18]-[20]. The way in which the components and the system safety states change is illustrated in *Figure 1.*

Definition 1. A vector

$$S_i(t,\cdot) = [S_i(t,0), S_i(t,1), ..., S_i(t,z)],\ t \in <0,\infty),\ i = 1,2,...,n, \tag{1}$$

where

$$S_i(t,u) = P(s_i(t) \geq u \mid s_i(0) = z) = P(T_i(u) > t),\ t \in <0,\infty),\ u = 0,1,...,z, \tag{2}$$

is the probability that the component E_i is in the safety state subset $\{u, u+1, ..., z\}$ at the moment t, $t \in <0,\infty)$, while it was in the safety state z at the moment $t = 0$, is called the multi-state safety function of a component E_i.

The safety functions $S_i(t,u)$, $t \in <0,\infty)$, $u = 0,1,...,z$, defined by (2) are called the coordinates of the component E_i, $i = 1,2,...,n$, multistate safety function $S_i(t,\cdot)$ given by (1). Thus, the relationship between the distribution function $F_i(t,u)$ of the component E_i, $i = 1,2,...,n$, lifetime $T_i(u)$ in the safety state subset $\{u, u+1, ..., z\}$ and the coordinate $S_i(t,u)$ of its multistate safety function is given by

$$F_i(t,u) = P(T_i(u) \leq t) = 1 - P(T_i(u) > t) = 1 - S_i(t,u),\ t \in <0,\infty),\ u = 0,1,...,z.$$

Under *Definition 1* and the agreements, we have the following property of the component multistate safety function coordinates

$$S_i(t,0) \geq S_i(t,1) \geq \ldots \geq S_i(t,z),\ t \in <0,\infty),\ i = 1,2,...,n.$$

Further, if we denote by

$$p_i(t,u) = P(s_i(t) = u \mid s_i(0) = z),\ t \in <0,\infty),\ u = 0,1,...,z,$$

the probability that the component E_i is in the safety state u at the moment t, while it was in the safety state z at the moment $t = 0$, then by (1)

$$S_i(t,0) = 1,\ S_i(t,z) = p_i(t,z),\ t \in <0,\infty),\ i = 1,2,...,n, \tag{3}$$

and

$$p_i(t,u) = S_i(t,u) - S_i(t,u+1),\ u = 0,1,...,z-1,\ t \in <0,\infty),\ i = 1,2,...,n. \tag{4}$$

Moreover, if

$$S_i(t,u) = 1 \text{ for } t \leq 0,\ u = 1,2,...,z,\ i = 1,2,...,n,$$

then

$$\mu_i(u) = \int_0^\infty S_i(t,u)dt,\ u = 1,2,...,z,\ i = 1,2,...,n, \tag{5}$$

is the mean lifetime of the component E_i in the safety state subset $\{u, u+1, ..., z\}$,

$$\sigma_i(u) = \sqrt{n_i(u) - [\mu_i(u)]^2},\ u = 1,2,...,z,\ i = 1,2,...,n, \tag{6}$$

where

$$n_i(u) = 2\int_0^\infty t S_i(t,u)dt,\ u = 1,2,...,z,\ i = 1,2,...,n, \tag{7}$$

is the standard deviation of the component E_i lifetime in the safety state subset $\{u, u+1, ..., z\}$ and

$$\bar{\mu}_i(u) = \int_0^\infty p_i(t,u)dt,\ u = 1,2,...,z,\ i = 1,2,...,n, \tag{8}$$

is the mean lifetime of the component E_i in the safety state u, in the case when the integrals defined by (5), (7) and (8) are convergent.

Next, according to (3), (4), (5) and (8), we have

$$\bar{\mu}_i(u) = \mu_i(u) - \mu_i(u+1),\ u = 0,1,...,z-1,$$
$$\bar{\mu}_i(z) = \mu_i(z),\ i = 1,2,...,n. \tag{9}$$

Definition 2. A vector

$$\boldsymbol{S}(t,\cdot) = [\boldsymbol{S}(t,0), \boldsymbol{S}(t,1), ..., \boldsymbol{S}(t,z)],\ t \in <0,\infty), \tag{10}$$

where

$$\boldsymbol{S}(t,u) = P(S(t) \geq u \mid S(0) = z) = P(T(u) > t), \quad (11)$$
$$t \in <0,\infty), \; u = 0,1,...,z,$$

is the probability that the system is in the safety state subset $\{u, u+1,..., z\}$ at the moment t, $t \in <0,\infty)$, while it was in the safety state z at the moment $t = 0$, is called the multi-state safety function of this system.

The safety functions $\boldsymbol{S}(t,u)$, $t \in <0,\infty)$, u = 0,1,...,z, defined by (11) are called the coordinates of the system multistate safety function $\boldsymbol{S}(t,\cdot)$ given by (10). Consequently, the relationship between the distribution function $\boldsymbol{F}(t,u)$ of the system S lifetime $T(u)$ in the safety state subset $\{u, u+1,..., z\}$ and the coordinate $\boldsymbol{S}(t,u)$ of its multistate safety function is given by

$$\boldsymbol{F}(t,u) = P(T(u) \leq t) = 1 - P(T(u) > t) = 1 - \boldsymbol{S}(t,u),$$
$$t \in <0,\infty), \; u = 0,1,...,z.$$

The exemplary graph of a four-state (z = 3) system safety function

$$\boldsymbol{S}(t,\cdot) = [1, \boldsymbol{S}(t,1), \boldsymbol{S}(t,2), \boldsymbol{S}(t,3)], \; t \in <0,\infty),$$

is shown in *Figure 2*.

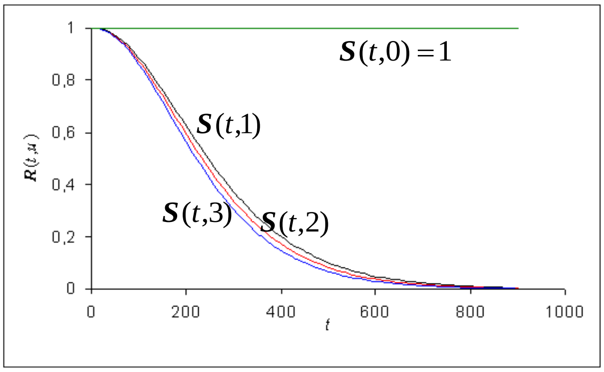

Figure 2. The graph of a four-state system safety function $\boldsymbol{S}(t,\cdot)$ coordinates

Under *Definition 2,* we have

$$\boldsymbol{S}(t,0) \geq \boldsymbol{S}(t,1) \geq \ldots \geq \boldsymbol{S}(t,z), \; t \in <0,\infty),$$

and if

$$p(t,u) = P(S(t) = u \mid S(0) = z), \; t \in <0,\infty), \quad (12)$$
$$u = 0,1,...,z,$$

is the probability that the system is in the safety state u at the moment t, $t \in <0,\infty)$, while it was in the safety state z at the moment t = 0, then

$$\boldsymbol{S}(t,0) = 1, \; \boldsymbol{S}(t,z) = p(t,z), \; t \in <0,\infty), \quad (13)$$

and

$$p(t,u) = \boldsymbol{S}(t,u) - \boldsymbol{S}(t,u+1), \; u = 0,1,..., z-1, \quad (14)$$
$$t \in <0,\infty).$$

Moreover, if

$$\boldsymbol{S}(t,u) = 1 \text{ for } t \leq 0, \; u = 1,2,...,z,$$

then

$$\mu(u) = \int_0^\infty \boldsymbol{S}(t,u)dt, \; u = 1,2,...,z, \quad (15)$$

is the mean lifetime of the system in the safety state subset $\{u, u+1,..., z\}$,

$$\sigma(u) = \sqrt{n(u) - [\mu(u)]^2}, \; u = 1,2,...,z, \quad (16)$$

where

$$n(u) = 2\int_0^\infty t\,\boldsymbol{S}(t,u)dt, \; u = 1,2,...,z, \quad (17)$$

is the standard deviation of the system lifetime in the safety state subset $\{u, u+1,..., z\}$ and moreover

$$\bar{\mu}(u) = \int_0^\infty p(t,u)dt, \; u = 1,2,...,z, \quad (18)$$

is the mean lifetime of the system in the safety state u while the integrals (15), (17) and (18) are convergent.

Additionally, according to (13), (14), (15) and (18), we get the following relationship

$$\bar{\mu}(u) = \mu(u) - \mu(u+1), \; u = 0,1,..., z-1,$$
$$\bar{\mu}(z) = \mu(z). \quad (19)$$

Definition 3. A probability

$$\boldsymbol{r}(t) = P(S(t) < r \mid S(0) = z) = P(T(r) \leq t), \; t \in <0,\infty),$$

that the system is in the subset of safety states worse than the critical safety state r, $r \in \{1,...,z\}$ while it was in the safety state z at the moment t = 0 is called

a risk function of the multi-state system [6], [11], [14].
Under this definition, from (11), we have

$$\boldsymbol{r}(t) = 1 - P(S(t) \geq r \mid S(0) = z) = 1 - \boldsymbol{S}(t,r), \quad t \in <0,\infty), \qquad (20)$$

and if τ is the moment when the system risk exceeds a permitted level δ, then

$$\tau = \boldsymbol{r}^{-1}(\delta), \qquad (21)$$

where $\boldsymbol{r}^{-1}(t)$, if it exists, is the inverse function of the system risk function $\boldsymbol{r}(t)$.
The exemplary graph of a four-state system risk function for the critical safety state $r = 2$

$$\boldsymbol{r}(t) = 1 - \boldsymbol{S}(t,2), \ t \in <0,\infty),$$

corresponding to the safety function illustrated in *Figure 2* is shown in *Figure 3.*

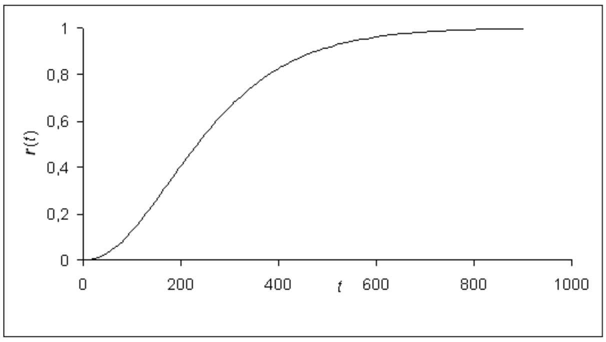

Figure 3. The graph of a four-state system risk function $\boldsymbol{r}(t)$

3. Safety structures of multistate systems

Now, after introducing the notion of the multistate safety analysis, we may define basic multi-state safety structures.

Definition 4. A multistate system is called series if its lifetime $T(u)$ in the safety state subset $\{u, u+1,..., z\}$ is given by

$$T(u) = \min_{1 \leq i \leq n}\{T_i(u)\}, \ u = 1,2,...,z.$$

The number n is called the system structure shape parameter.
The above definition means that a multistate series system is in the safety state subset $\{u, u+1,..., z\}$ if and only if all its n components are in this subset of safety states. That meaning is very close to the definition of a two-state series system considered in a classical reliability [11], [14] analysis that is not failed if all its components are not failed. This fact can justify the safety structure scheme for a multistate series system presented in *Figure 4.*

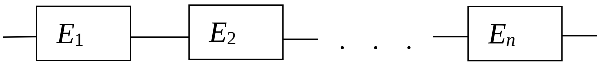

Figure 4. The scheme of a series system safety structure

It is easy to work out that the safety function of the multistate series system is given by the vector [6], [11]-[15]

$$\boldsymbol{S}(t,\cdot) = [1, \boldsymbol{S}(t,1), ..., \boldsymbol{S}(t,z)] \qquad (22)$$

with the coordinates

$$\boldsymbol{S}(t,u) = \prod_{i=1}^{n} S_i(t,u), \ t \in <0,\infty), \ u = 1,2,...,z. \qquad (23)$$

Definition 5. A multistate system is called parallel if its lifetime $T(u)$ in the safety state subset $\{u, u+1,..., z\}$ is given by

$$T(u) = \max_{1 \leq i \leq n}\{T_i(u)\}, \ u = 1,2,...,z.$$

The number n is called the system structure shape parameter.
The above definition means that the multistate parallel system is in the safety state subset $\{u, u+1,..., z\}$ if and only if at least one of its n components is in this subset of safety states. That meaning is very close to the definition of a two-state parallel system in a classical reliability [11], [14] analysis that is not failed if at least one of its components is not failed what can justify the safety structure scheme for a multistate parallel system presented in *Figure 5.*

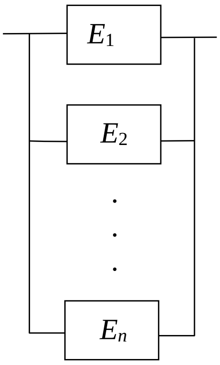

Figure 5. The scheme of a parallel system safety structure

The safety function of the multistate parallel system is given by the vector [6], [11], [14]

$$\boldsymbol{S}(t,\cdot) = [1,\boldsymbol{S}(t,1),...,\boldsymbol{S}(t,z)],$$

with the coordinates

$$\boldsymbol{S}(t,u) = 1 - \prod_{i=1}^{n} F_i(t,u), \ t \in <0,\infty), \ u = 1,2,...,z.$$

Definition 6. A multistate system is called an "*m* out of *n*" system if its lifetime $T(u)$ in the safety state subset $\{u,u+1,...,z\}$ is given by

$$T(u) = T_{(n-m+1)}(u), \ m = 1,2,...,n, \ u = 1,2,...,z,$$

where $T_{(n-m+1)}(u)$ is the *m*-th maximal order statistic in the sequence of the component lifetimes

$$T_1(u), T_2(u),..., T_n(u), \ u = 1,2,...,z.$$

The above definition means that the multistate „*m* out of *n*" system is in the safety state subset $\{u,u+1,...,z\}$ if and only if at least *m* out of its *n* components are in this safety state subset and it is a multistate parallel system if *m* = 1 and it is a multistate series system if *m* = *n*. The numbers *m* and *n* are called the system structure shape parameters. The scheme of an "*m* out of *n*" multistate system safety structure, justified in an analogous way as in the case of a multistate series system and a multistate parallel system, is given in *Figure 6*, where i_1, i_2, …, $i_n \in \{1,2,...,n\}$ and $i_a \neq i_b$ for $a \neq b$.

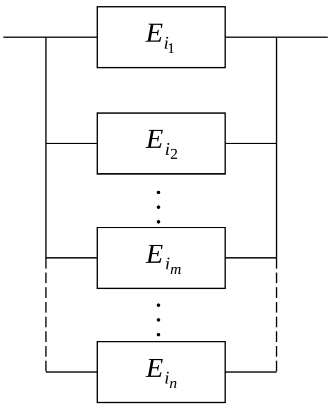

Figure 6. The scheme of an "*m* out of *n*" system safety structure

It can be simply shown that the safety function of the multistate "*m* out of *n*" system is given either by the vector [6], [11], [14]

$$\boldsymbol{S}(t,\cdot) = [1, \boldsymbol{S}(t,1),..., \boldsymbol{S}(t,z)],$$

with the coordinates

$$\boldsymbol{S}(t,u) = 1 - \sum_{\substack{r_1,r_2,...,r_n=0 \\ r_1+r_2+...+r_n \le m-1}}^{1} [S_i(t,u)]^{r_i} [F_i(t,u)]^{1-r_i},$$
$$t \in <0,\infty), \ u = 1,2,...,z,$$

or by the vector [6], [11], [14]

$$\boldsymbol{S}(t,\cdot) = [1, \boldsymbol{S}(t,1),..., \boldsymbol{S}(t,z)],$$

with the coordinates

$$\boldsymbol{S}(t,u) = \sum_{\substack{r_1,r_2,...,r_n=0 \\ r_1+r_2+...+r_n \le \bar{m}}}^{1} [F_i(t,u)]^{r_i} [S_i(t,u)]^{1-r_i},$$
$$t \in <0,\infty), \ \bar{m} = n - m, \ u = 1,2,...,z.$$

Definition 7. A multistate system is called a consecutive "*m* out of *n*: F" system if it is out of the safety state subset $\{u,u+1,...,z\}$ if and only if at least its *m* neighbouring components out of *n* its components arranged in a sequence of E_1, E_2, ..., E_n, are out of this safety state subset. The numbers *m* and *n* are called the system structure shape parameters.
After denoting by

$$\boldsymbol{S}(t,u) = P(S(t) \ge u \,|\, S(0) = z) = P(T(u) > t),$$
$$t \in <0,\infty), \ u = 0,1,...,z,$$

the probability that the consecutive "*m* out of *n*: F" system is in the safety state subset $\{u,u+1,...,z\}$ at the moment *t*, $t \in <0,\infty)$, while it was in the safety state *z* at the moment *t* = 0 and by

$$\boldsymbol{F}(t,u) = P(T(u) \le t), \ t \in <0,\infty), \ u = 0,1,...,z,$$

the distribution function of the lifetime $T(u)$ of this system in the safety state subset $\{u,u+1,...,z\}$, while it was in the safety state *z* at the moment *t* = 0, we conclude that the safety function of the consecutive "*m* out of *n*: F" system is the given by the vector

$$\boldsymbol{S}(t,\cdot) = [1, \ \boldsymbol{S}(t,1), \ \boldsymbol{S}(t,2), \ ..., \ \boldsymbol{S}(t,z)],$$

with the coordinates given by the following recurrent formula [6], [11], [14]

$$\boldsymbol{S}(t,u) = $$

$$= \boldsymbol{S}_n(t,u) = \begin{cases} 1 & \text{for } n < m, \\ 1 - \prod_{i=1}^{n} F_i(t,u) & \text{for } n = m, \\ S_n(t,u)\boldsymbol{S}_{n-1}(t,u) & \\ + \sum_{i=1}^{m-1} S_{n-i}(t,u)\boldsymbol{S}_{n-i-1}(t,u) & \\ \cdot \prod_{j=n-i+1}^{n} F_j(t,u) & \text{for } n > m, \end{cases}$$

for $t \in <0,\infty)$, $u = 1,2,...,z$.

Other basic multistate safety structures with components degrading in time are series-parallel, parallel-series, series-"m out of k", "m_i out of l_i"-series, series-consecutive "m out of k: F" and consecutive "m_i out of l_i: F"-series systems.
To define them, we assume that:

– k is the number of the system subsystems,
– l_i, $i = 1,2,...,k$, are the numbers of the subsystem components,
– E_{ij}, $i = 1,2,...,k$, $j = 1,2,...,l_i$, $k, l_1, l_2,..., l_k \in N$, are components of a system,
– all components E_{ij} have the same safety state set as before $\{0,1,...,z\}$,
– $T_{ij}(u)$, $i = 1,2,...,k$, $j = 1,2,...,l_i$, $k, l_1, l_2,..., l_k \in N$, are independent random variables representing the lifetimes of components E_{ij} in the safety state subset $\{u,u+1,...,z\}$, while they were in the safety state z at the moment $t = 0$,
– $E_{ij}(t)$ is a component E_{ij} safety state at the moment t, $t \in <0,\infty)$, while they were in the safety state z at the moment $t = 0$,

Definition 8
A vector

$$S_{ij}(t,\cdot) = [S_{ij}(t,0), S_{ij}(t,1),..., S_{ij}(t,z)], \quad (24)$$
$$t \in <0,\infty), \; i = 1,2,...,k, \; j = 1,2,...,l_i,$$

where

$$S_{ij}(t,u) = P(E_{ij}(t) \geq u \mid E_{ij}(0) = z) \quad (25)$$
$$= P(T_{ij}(u) > t),$$
$$t \in <0,\infty), \; u = 0,1,...,z,$$

is the probability that the component E_{ij} is in the safety state subset $\{u,u+1,...,z\}$ at the moment t, $t \in <0,\infty)$, while it was in the safety state z at the moment $t = 0$, is called the safety function of a multistate component E_{ij}.

The safety functions $S_{ij}(t,u)$, $t \in <0,\infty)$, $u = 0,1,...,z$, defined by (25) are called the coordinates of the component E_{ij}, $i = 1,2,...,k$, $j = 1,2,...,l_i$, safety function $S_{ij}(t,\cdot)$ given by (24). Thus, the relationship between the distribution function $F_{ij}(t,u)$ of the component E_{ij}, $i = 1,2,...,k$, $j = 1,2,...,l_i$, lifetime $T_{ij}(u)$ in the safety state subset $\{u,u+1,...,z\}$ and the coordinate $S_{ij}(t,u)$ of its safety function is given by

$$F_{ij}(t,u) = P(T_{ij}(u) \leq t) = 1 - P(T_{ij}(u) > t)$$
$$= 1 - S_{ij}(t,u), \; t \in <0,\infty), \; u = 0,1,...,z.$$

Definition 9
A multistate system is called series-parallel if its lifetime $T(u)$ in the state subset $\{u,u+1,...,z\}$ is given by

$$T(u) = \max_{1 \leq i \leq k}\{\min_{1 \leq j \leq l_i}\{T_{ij}(u)\}\}, \; u = 1,2,...,z.$$

The above definition means that the multistate series-parallel system is composed of k multistate series subsystems and it is in the safety state subset $\{u,u+1,...,z\}$ if and only if at least one out of its k series subsystems is in this safety state subset. In this definition, l_i, $i = 1,2,...,k$, denote the numbers of components in the series subsystems. The numbers k and l_1, l_2,..., l_k are called the system structure shape parameters. Joining the justification for the safety structure schemes of the multistate series system and the multistate parallel system leads to the scheme of a multistate series-parallel system safety structure given in *Figure 7*.

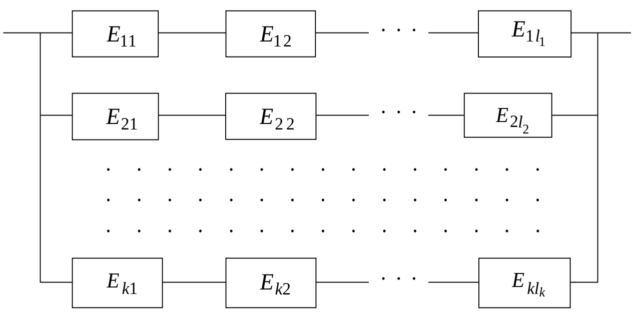

Figure 7. The scheme of a series-parallel system safety structure

The safety function of the multistate series-parallel system is given by the vector [6], [11], [14]

$$\boldsymbol{S}(t,\cdot) = [1, \; \boldsymbol{S}(t,1), \; \boldsymbol{S}(t,2), \; ..., \; \boldsymbol{S}(t,z)], \quad (26)$$

with the coordinates

$$\boldsymbol{S}(t,u) = 1-\prod_{i=1}^{k}[1-\prod_{j=1}^{l_i} S_{ij}(t,u)],\ t \in <0,\infty), \quad (27)$$

$u = 1,2,...,z,$

where k is the number of series subsystems linked in parallel and l_i are the numbers of components in the series subsystems.

Proceed in analogous way as before we define other basic multistate safety structures [6]. All multistate safety structures are defined in [6].

Proposition 1

If components of the multistate system have the exponential safety functions [6], [11], [14]

$$S_i(t,\cdot) = [1, S_i(t,1), \ldots, S_i(t,z)],\ t \in (-\infty,\infty),$$

where

$$S_i(t,u) = 1, \text{ for } t < 0,$$
$$S_i(t,u) = \exp[-\lambda_i(u)t] \text{ for } t \geq 0,$$
$$\lambda_i(u) > 0,\ i = 1,2,...,n,\ u = 1,2,\ldots,z,$$

in the case of the multistate series, parallel, "m out of n", consecutive "m out of n" systems and respectively

$$S_{ij}(t,\cdot) = [1, S_{ij}(t,1), \ldots, S_{ij}(t,z)],\ t \in (-\infty,\infty),$$

where

$$S_{ij}(t,u) = 1 \text{ for } t < 0,$$
$$S_{ij}(t,u) = \exp[-\lambda_{ij}(u)t] \text{ for } t \geq 0,$$
$$\lambda_{ij}(u) > 0,\ i = 1,2,...,n,\ j = 1,2,...,l_i,\ u = 1,2,\ldots,z,$$

in the case of the series-parallel, parallel-series, series-"m out of k", "m_i out of l_i"-series, series-consecutive "m out of k" and consecutive "m_i out of l_i"-series systems, then its safety function is given by the vector:

i) for a series system

$$\boldsymbol{S}(t,\cdot) = [1,\ \boldsymbol{S}(t,1),\ \boldsymbol{S}(t,2),\ ...,\ \boldsymbol{S}(t,z)], \quad (28)$$

where

$$\boldsymbol{S}(t,u) = 1 \text{ for } t < 0,\ \boldsymbol{S}(t,u) = \exp[-\sum_{i=1}^{n}\lambda_i(u)t] \quad (29)$$

for $t \geq 0,\ u = 1,2,...,z,$

ii) for a parallel system

$$\boldsymbol{S}(t,\cdot) = [1,\ \boldsymbol{S}(t,1),\ \boldsymbol{S}(t,2),\ ...,\ \boldsymbol{S}(t,z)],$$

where

$$\boldsymbol{S}(t,u) = 1 \text{ for } t < 0,$$
$$\boldsymbol{S}(t,u) = 1-\prod_{i=1}^{n}[1-\exp[-\lambda_i(u)t]], \text{ for } t \geq 0,$$
$$u = 1,2,...,z,$$

iii) for a "m out of n" system

$$\boldsymbol{S}(t,\cdot) = [1,\ \boldsymbol{S}(t,1),\ \boldsymbol{S}(t,2),\ ...,\ \boldsymbol{S}(t,z)],$$

where

(2.54)

$$\boldsymbol{S}(t,u) = 1 \text{ for } t < 0,$$
$$\boldsymbol{S}(t,u) = 1-\sum_{\substack{r_1,r_2,...,r_n=0\\ r_1+r_2+...+r_n \leq m-1}}^{1}\prod_{i=1}^{n}\exp[-r_i\lambda_i(u)t][1-\exp[-\lambda_i(u)t]]^{1-r_i}$$

for $t \in <0,\infty)$, $u = 1,2,...,z$, or

$$\boldsymbol{S}(t,\cdot) = [1,\ \boldsymbol{S}(t,1),\ \boldsymbol{S}(t,2),\ ...,\ \boldsymbol{S}(t,z)],$$

where

(2.56)

$$\boldsymbol{S}(t,u) = 1 \text{ for } t < 0,$$

$$\boldsymbol{S}(t,u) = \sum_{\substack{r_1,r_2,...,r_n=0\\ r_1+r_2+...+r_n \leq \bar{m}}}^{1}\prod_{i=1}^{n}[1-\exp[-\lambda_i(u)t]]^{r_i}\exp[-(1-r_i)\lambda_i(u)t]$$

for $t \in <0,\infty)$, $\bar{m} = n-m$, $u = 1,2,...,z$,

iv) for a consecutive "m out of n: F" system

$$\boldsymbol{S}(t,\cdot) = [1,\ \boldsymbol{S}(t,1),\ \boldsymbol{S}(t,2),\ ...,\ \boldsymbol{S}(t,z)],$$

where

$$\boldsymbol{S}(t,u) = 1 \text{ for } t < 0,$$

$$\boldsymbol{S}(t,u) = \boldsymbol{S}_n(t,u)$$

$$= \begin{cases} 1 & \text{for } n < m, \\ 1 - \prod_{i=1}^{n}[1 - \exp[-\lambda_i(u)t]] & \text{for } n = m, \\ \exp[-\lambda_n(u)t]\boldsymbol{S}_{n-1}(t,u) & \\ \quad + \sum_{i=1}^{m-1}\exp[-\lambda_{n-i}(u)t]\boldsymbol{S}_{n\text{-}i\text{-}1}(t,u) & \\ \cdot \prod_{j=n-i+1}^{n}[1 - \exp[-\lambda_j(u)t]] & \text{for } n > m, \end{cases}$$

for $t \in <0,\infty)$, $u = 1,2,...,z$,

v) for a series-parallel system

$$\boldsymbol{S}(t,\cdot) = [1,\ \boldsymbol{S}(t,1),\ \boldsymbol{S}(t,2),\ ...,\ \boldsymbol{S}(t,z)\,], \tag{30}$$

where

$$\boldsymbol{S}(t,u) = 1 \text{ for } t < 0$$

$$\boldsymbol{S}(t,u) = 1 - \prod_{i=1}^{k}[1 - \exp[-\sum_{j=1}^{l_i}\lambda_{ij}(u)t]] \text{ for } t \geq 0, \tag{31}$$

$$u = 1,2,...,z.$$

The safety functions for other multistate systems, such as parallel-series, series-" m out of k ", "m_i out of l_i"-series, series-consecutive " m out of k " and consecutive "m_i out of l_i"-series are given in [6].

5. Application

The considered oil piping transportation system is operating at one of the Baltic Oil Terminals that is designated for the reception from ships, the storage and sending by carriages or cars the oil products. It is also designated for receiving from carriages or cars, the storage and loading the tankers with oil products such like petrol and oil. The considered terminal is composed of three parts *A*, *B* and *C*, linked by the piping transportation system with the pier. The scheme of this terminal is presented in *Figure 8* [5].

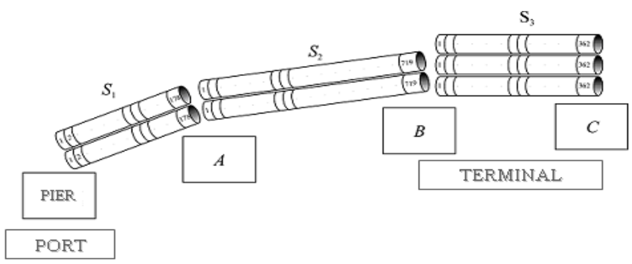

Figure 8. The scheme of the port oil piping transportation system

Thus, the port oil pipeline transportation system consists of three subsystems:

- the subsystem S_1 composed of two pipelines, each composed of 178 pipe segments and 2 valves,
- the subsystem S_2 composed of two pipelines, each composed of 717 pipe segments and 2 valves,
- the subsystem S_3 composed of three pipelines, each composed of 360 pipe segments and 2 valves.

The subsystems S_1, S_2, S_3, indicated in *Figure 8* are forming a general series port oil pipeline system safety structure presented in *Figure 9*.

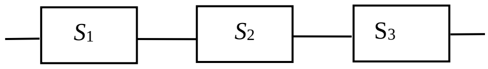

Figure 9. General scheme of the port oil pipeline system safety structure

The system is a series system composed of two series-parallel subsystems S_1, S_2, each containing two pipelines and one series-"2 out of 3" subsystem S_3, containing 3 pipelines.

The subsystems S_1, S_2 and S_3 are forming a general series port oil pipeline system safety structure presented in *Figure 9*.

After considering the comments and opinions coming from experts, taking into account the effectiveness and safety aspects of the operation of the oil pipeline transportation system, we distinguish the following three safety states $(z = 2)$ of the system and its components:

- a safety state 2 – piping operation is fully safe,
- a safety state 1 – piping operation is less safe and more dangerous because of the possibility of environment pollution,
- a safety state 0 – piping is destroyed.

Moreover, by the expert opinions, we assume that there are possible the transitions between the components safety states only from better to worse ones and we assume that the system and its components critical safety state is $r = 1$.

From the above, the subsystems S_υ, $\upsilon = 1,2,3$, are composed of three-state, i.e. $z = 2$, components $E_{ij}^{(\upsilon)}$, $\upsilon = 1,2,3,$, having the safety functions

$$S_{ij}^{(\upsilon)}(t,\cdot) = [1,\ S_{ij}^{(\upsilon)}(t,1), S_{ij}^{(\upsilon)}(t,2)\,],$$

with the coordinates that by the assumption are exponential of the forms

$$S_{ij}^{(\upsilon)}(t,1) = \exp[-\lambda_{ij}^{(\upsilon)}(1)t],\ \ S_{ij}^{(\upsilon)}(t,2) = \exp[-\lambda_{ij}^{(\upsilon)}(2)t].$$

Thus, the system is a three-state series system composed of two three-state series-parallel subsystems S_1, S_2, each containing two pipelines and one three-state series-"2 out of 3" subsystem S_3. The subsystem S_1 consists of k = 2 identical pipelines, each composed of 178 components $E_{ij}^{(1)}$, $i=1,2,\ j=1,2,...,178$, i.e. $l_1 = l_2 = 178$, with the exponential safety functions identified on the basis of data coming from experts and given below.

In each pipeline there are:

- 176 pipe segments with the multistate safety functions co-ordinates

$$S_{ij}^{(1)}(t,1) = \exp[-0.0062t],$$
$$S_{ij}^{(1)}(t,2) = \exp[-0.0088t],\ i=1,2,\ j=1,2,...,176,$$

- 2 valves with the multistate safety functions co-ordinates

$$S_{ij}^{(1)}(t,1) = \exp[-0.0167t],$$
$$S_{ij}^{(1)}(t,2) = \exp[-0.0182t],\ i=1,2,\ j=177,178.$$

The subsystem S_1 is a three-state series-parallel system and according to (26)-(27) and (30)-(31) its three-state safety function is given by

$$\boldsymbol{S}^{(1)}(t,\cdot) = [1,\ \boldsymbol{S}^{(1)}(t,1),\ \boldsymbol{S}^{(1)}(t,2)],\ t \geq 0, \tag{32}$$

where

$$\boldsymbol{S}^{(1)}(t,1) = 1-\prod_{i=1}^{2}[1-\prod_{j=1}^{178} R_{ij}^{(1)}(t,1)]$$
$$= 1-[1-\exp[-[176\cdot 0.0062+2\cdot 0.0167]t]]^2$$
$$= 1-[1-\exp[-1.1246t]]^2$$
$$= 2\exp[-1.1246t]-\exp[-2.2492t], \tag{33}$$
$$\boldsymbol{S}^{(1)}(t,2) = 1-\prod_{i=1}^{2}[1-\prod_{j=1}^{178} R_{ij}^{(1)}(t,2)]$$
$$= 1-[1-\exp[-[176\cdot 0.0088+2\cdot 0.0182]t]]^2$$
$$= 1-[1-\exp[-1.5852t]]^2$$
$$= 2\exp[-1.5852t]-\exp[-3.1704t]. \tag{34}$$

The subsystem S_2 consists of k = 2 identical pipelines, each composed of 719 components $E_{ij}^{(2)}$, $i=1,2,\ j=1,2,...,719$, i.e. $l_1 = l_2 = 719$, with the exponential safety functions identified on the basis of data coming from experts and given below. In each pipeline there are:

- 717 pipe segments with the multistate safety functions co-ordinates

$$S_{ij}^{(2)}(t,1) = \exp[-0.0062t],$$
$$S_{ij}^{(2)}(t,2) = \exp[-0.0088t],\ i=1,2,\ j=1,2,...,717,$$

- 2 valves with the multistate safety functions co-ordinates

$$S_{ij}^{(2)}(t,1) = \exp[-0.0166t],$$
$$S_{ij}^{(2)}(t,2) = \exp[-0.0181t],\ i=1,2,\ j=718,719.$$

Thus, the subsystem S_2 is a three-state series-parallel system and according to (26)-(27) and (30)-(31) its three-state safety function is given by

$$\boldsymbol{S}^{(2)}(t,\cdot) = [1,\ \boldsymbol{S}^{(2)}(t,1),\ \boldsymbol{S}^{(2)}(t,2)],\ t \geq 0, \tag{35}$$

where

$$\boldsymbol{S}^{(2)}(t,1) = 1-\prod_{i=1}^{2}[1-\prod_{j=1}^{719} R_{ij}^{(2)}(t,1)]$$
$$= 1-[1-\exp[-[717\cdot 0.0062+2\cdot 0.0166]t]]^2$$
$$= 1-[1-\exp[-4.4786t]]^2$$
$$= 2\exp[-4.4786t]-\exp[-8.9572t], \tag{36}$$
$$\boldsymbol{S}^{(2)}(t,2) = 1-\prod_{i=1}^{2}[1-\prod_{j=1}^{719} R_{ij}^{(2)}(t,2)]$$
$$= 1-[1-\exp[-[717\cdot 0.0088+2\cdot 0.0181]t]]^2$$
$$= 1-[1-\exp[-6.3458t]]^2$$
$$= 2\exp[-6.3458\,t]-\exp[-12.6916\,t]. \tag{37}$$

The subsystem S_3 consists of k = 3 pipelines, two pipelines of the first type and one pipeline of the second type, each composed of l = 362 components $E_{ij}^{(3)}$, $i=1,2,3,\ j=1,2,...,362$, i.e. $l_1 = l_2 = l_3 = 362$, with the exponential safety functions identified on the basis of data coming from experts and given below. In each pipeline of the first type there are:

- 360 pipe segments with the multistate safety functions co-ordinates

$$S_{ij}^{(3)}(t,1) = \exp[-0.0059t],$$
$$S_{ij}^{(3)}(t,2) = \exp[-0.0074t],\ i=1,2,\ j=1,2,...,360,$$

- 2 valves with the multistate safety functions co-ordinates

$$S_{ij}^{(3)}(t,1) = \exp[-0.0166t],$$
$$S_{ij}^{(3)}(t,2) = \exp[-0.0181t],\ i=1,2,\ j=361,362.$$

In the pipeline of the second type there are:

- 360 pipe segments with the multistate safety functions co-ordinates

$$S_{ij}^{(3)}(t,1) = \exp[-0.0071t],$$

$S_{ij}^{(3)}(t,2) = \exp[-0.0079t],\ i = 3\ \ j = 1,2,...,360,$

- 2 valves with the multistate safety functions co-ordinates

$S_{ij}^{(3)}(t,1) = \exp[-0.0166t],$

$S_{ij}^{(3)}(t,2) = \exp[-0.0181t],\ i = 3,\ j = 361,362.$

The subsystem S_3 is a three-state series-"2 out of 3" system and according to (2.40)-(2.41) and (2.72)-(2.73) in [6] its multistate safety function is given by

$$\boldsymbol{S}^{(3)}(t,\cdot) = [1,\ \boldsymbol{S}^{(3)}(t,1),\ \boldsymbol{S}^{(3)}(t,2)],\ t \geq 0, \quad (38)$$

where

$$\begin{aligned}
&\boldsymbol{S}^{(3)}(t,1) \\
&= 1 - \sum_{\substack{r_1,r_2,r_3=0\\ r_1+r_2+r_3\leq 1}}^{1} \prod_{i=1}^{3} [\prod_{j=1}^{362} R_{ij}^{(3)}(t,1)]^{r_i} [1 - \prod_{j=1}^{362} R_{ij}^{(3)}(t,1)]^{1-r_i} \\
&= 1 - [1 - \exp[-[360 \cdot 0.0059 + 2 \cdot 0.0166]t]]^2 \\
&\cdot [1 - \exp[-[360 \cdot 0.0071 + 2 \cdot 0.0166]t]]^1 \\
&- 2\exp[-1[360 \cdot 0.0059 + 2 \cdot 0.0166]t] \\
&[1 - \exp[-[360 \cdot 0.0059 + 2 \cdot 0.0166]t]]^1 \\
&\cdot [1 - \exp[-[360 \cdot 0.0071 + 2 \cdot 0.0166]t]]^1 \\
&- \exp[-[360 \cdot 0.0071 + 2 \cdot 0.0166]t] \\
&[1 - \exp[-[360 \cdot 0.0059 + 2 \cdot 0.0166]t]]^2 \\
&= \exp[-4.3144t] + 2\exp[-4.7464t] \\
&- 2\exp[-6.9036t], \quad (39)
\end{aligned}$$

$$\begin{aligned}
&\boldsymbol{S}^{(3)}(t,2) \\
&= 1 - \sum_{\substack{r_1,r_2,r_3=0\\ r_1+r_2+r_3\leq 1}}^{1} \prod_{i=1}^{3} [\prod_{j=1}^{362} S_{ij}^{(3)}(t,2)]^{r_i} [1 - \prod_{j=1}^{362} S_{ij}^{(3)}(t,2)]^{1-r_i} \\
&= 1 - [1 - \exp[-[360 \cdot 0.0074 + 2 \cdot 0.0181]t]]^2 \\
&\cdot [1 - \exp[-[360 \cdot 0.0079 + 2 \cdot 0.0181]t]]^1 \\
&- 2\exp[-1[360 \cdot 0.0074 + 2 \cdot 0.0181]t] \\
&[1 - \exp[-[360 \cdot 0.0074 + 2 \cdot 0.0181]t]]^1 \\
&\cdot [1 - \exp[-[360 \cdot 0.0079 + 2 \cdot 0.0181]t]]^1 \\
&- \exp[-[360 \cdot 0.0079 + 2 \cdot 0.0181]t] \\
&[1 - \exp[-[360 \cdot 0.0074 + 2 \cdot 0.0181]t]]^2 \\
&= \exp[-5.4004t] + 2\exp[-5.5804t] \\
&- 2\exp[-8.2806t]. \quad (40)
\end{aligned}$$

Considering that the pipeline system is a three-state series system, after applying (22)–(23), its safety function is given by

$$\boldsymbol{S}(t,\cdot) = [1, \boldsymbol{S}(t,1)\ \boldsymbol{S}(t,2)],\ t \geq 0, \quad (41)$$

where by (33)-(34), (36)-(37) and (39)-(40) we have

$$\begin{aligned}
&\boldsymbol{S}(t,1) = \boldsymbol{S}_3(t,1) = \boldsymbol{S}^{(1)}(t,1)\ \boldsymbol{S}^{(2)}(t,1)\ \boldsymbol{S}^{(3)}(t,1) \\
&= 4\exp[-9.9176t] + 8\exp[-10.3496t] \\
&- 8\exp[-12.5078t] - 2\exp[-14.396t] \\
&- 4\exp[-14.8282t] + 4\exp[-16.9864t] \\
&- 2\exp[-11.0422t] - 4\exp[-11,4742t] \\
&+ 4\exp[-13.6324t] + \exp[-15.5208t] \\
&+ 2\exp[-15.9528t] - 2\exp[-18.111t], \quad (42)
\end{aligned}$$

$$\begin{aligned}
&\boldsymbol{S}(t,2) = \bar{\boldsymbol{S}}_3(t,2) = \boldsymbol{S}^{(1)}(t,2)\ \boldsymbol{S}^{(2)}(t,2)\ \boldsymbol{S}^{(3)}(t,2) \\
&= 4\exp[-13.3314t] + 8\exp[-13.5114t] \\
&- 8\exp[-16.2116t] - 2\exp[-19.6772t] \\
&- 4\exp[-19.8572t] + 4\exp[-22.5574t] \\
&- 2\exp[-14.9166t] - 4\exp[-15,0966t] \\
&+ 4\exp[-17.7968t] + \exp[-21.2624t] \\
&+ 2\exp[-21.4424t] - 2\exp[-24.1426t]. \quad (43)
\end{aligned}$$

The graph of the three-state piping system safety function is shown in *Figure 10.*
The expected values and standard deviations of the pipeline system lifetimes in the safety state subsets $\{1,2\}$, $\{2\}$, calculated from the results given by (42)-(43), according to (15)-(17), respectively are:

$$\mu(1) \cong 0.207,\ \mu(2) \cong 0.156 \text{ year}, \quad (44)$$
$$\sigma(1) \cong 0.137,\ \sigma(2) \cong 0.104 \text{ year},$$

and further, using (44), by (19), it follows that the mean values of the piping lifetimes in the particular safety states are:

$$\bar{\mu}(1) \cong 0.051,\ \bar{\mu}(2) \cong 0.156 \text{ year}.$$

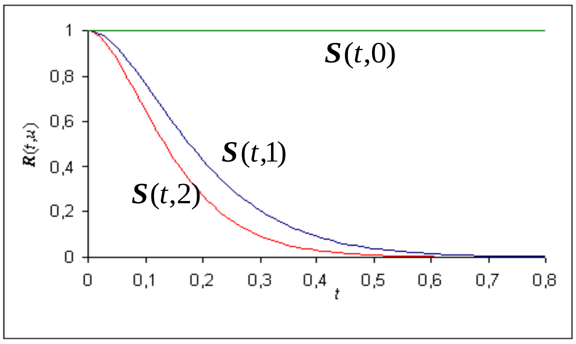

Figure 10. The graph of the port oil piping transportation system safety function $\boldsymbol{S}(t,\cdot)$ coordinates

As the critical safety state is r =1, then the pipeline system risk function, according to (20), is given by

$$\boldsymbol{r}(\mathrm{t}) = 1 - \boldsymbol{S}(t,1)$$

$$= 1–[\ 4\exp[-9.9176\,t] + 8\exp[-10.3496\,t]$$
$$-8\exp[-12.5078\,t]\ -2\exp[-14.396\,t]$$
$$-4\exp[-14.8282\,t] + 4\exp[-16.9864\,t]$$
$$-2\exp[-11.0422\,t] - 4\exp[-11{,}4742\,t]$$
$$+4\exp[-13.6324\,t]\ + \exp[-15.5208\,t]$$
$$+2\exp[-15.9528\,t] - 2\exp[-18.111t]] \text{ for } t \geq 0.$$

Hence, and from (21), the moment when the piping system risk function exceeds a permitted level, for instance $\delta = 0.05$, is

$$\tau = \boldsymbol{r}^{-1}(\delta) \cong 0.04.$$

The graph of the risk function $\boldsymbol{r}(t)$ of the three-state pipeline system is shown in *Figure 11*.

5. Conclusion

In the paper there is presented the safety model of ageing systems. Presented model is the basis for further considerations in particular tasks of the EU-CIRCLE project. Next this model together with the models of the system operation process presented in [5] will be used to construct the integrated general safety probabilistic model of the critical infrastructure related to its operation process.

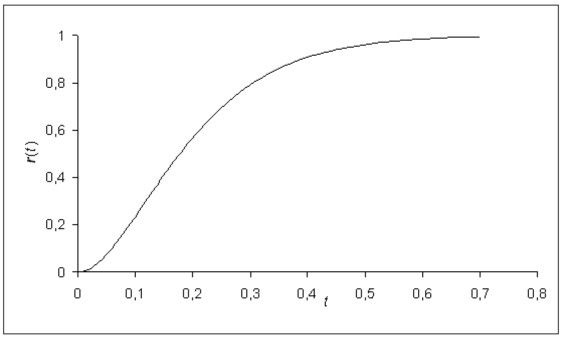

Figure 11. The graph of the risk function $\boldsymbol{r}(t)$ of the port oil piping transportation system

The models applied here, in their particular cases, for the safety analysis and prediction of the port oil piping transportation system operating in constant operation conditions will also be applied in tasks of the EU-CIRCLE project to safety analysis and prediction of these systems operating at the variable operation conditions.

Acknowledgments

The paper presents the results developed in the scope of the EU-CIRCLE project titled "A pan – European framework for strengthening Critical Infrastructure resilience to climate change" that has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 653824. http://www.eu-circle.eu/.

References

[1] Aven, T. (1985). Reliability evaluation of multistate systems with multistate components. *IEEE Transactions on Reliability* 34, 473-479.

[2] Aven, T. & Jensen, U. (1999). *Stochastic Models in Reliability*. Springer-Verlag, New York.

[3] Barlow, R. E. & Wu, A. S. (1978). Coherent systems with multi-state components. *Mathematics of Operations Research* 4, 275-281.

[4] Brunelle, R. D. & Kapur, K. C. (1999). Review and classification of reliability measures for multistate and continuum models. *IEEE Transactions on Reliability* 31, 1117-1180.

[5] EU-CIRCLE Report D1.1-GMU1. (2015). *EU-CIRCLE Taxonomy*.

[6] EU-CIRCLE Report D2.3-GMU1. (2016). *Identification methods and procedures of Critical Infrastructure Operation Process (CIOP) including Operating Environment Threats (OET).*

[7] EU-CIRCLE Report D3.3-GMU1. (2016). *Modelling inside dependences influence on safety of multistate ageing systems – Modelling safety of multistate ageing systems*.

[8] Huang, J., Zuo, M. J. & Wu, Y. (2000). Generalized multi-state k-out-of-n: G systems. *IEEE Transactions on Reliability* 49, 105-111.

[9] Hudson, J. C. & Kapur, K. C. (1983). Reliability analysis of multistate systems with multistate components. *IIE Transactions* 15, 127-135.

[10] Hudson, J. & Kapur, K. (1985). Reliability bounds for multistate systems with multistate components. *Operations Research* 33, 735-744.

[11] Kołowrocki, K. (2004). *Reliability of Large Systems.* Elsevier, Amsterdam - Boston - Heidelberg - London - New York - Oxford - Paris - San Diego - San Francisco - Singapore - Sydney - Tokyo.

[12] Kołowrocki, K. (2008). *Reliability of large systems.* In: Encyclopedia of Quantitative Risk Analysis and Assessment 4, John Wiley & Sons.

[13] Kolowrocki, K. &. Soszynska, J. (2011). On safety analysis of complex technical maritime transportation system. *Journal of Risk and Reliability* 225, 3, 345-354.

[14] Kołowrocki, K. & Soszyńska-Budny, J. (2011). *Reliability and safety of complex technical systems and processes: Modeling – Identification – Prediction – Optimization.* Springer, ISBN 978-0-85729-693-1.

[15] Kołowrocki, K. & Soszyńska-Budny, J. (2012). Preliminary approach to safety analysis of critical

infrastructures. Summer Safety and Reliability Seminars – SSARS 2012. *Journal of Polish Safety and Reliability Association* 3, 1, 73-88.

[16] Meng, F. (1993). Component-relevancy and characterisation in multi-state systems. *IEEE Transactions on Reliability* 42, 478-483.

[17] Soszyńska-Budny, J. (2014). Safety of complex technical systems, (in preparation).

[18] Xue, J. (1985). On multi-state system analysis. *IEEE Transactions on Reliability* 34, 329-337.

[19] Xue, J. & Yang, K. (1995). Dynamic reliability analysis of coherent multi-state systems. *IEEE Transactions on Reliability* 44, 683-688.

[20] Xue, J. & Yang, K. (1995). Symmetric relations in multi-state systems. *IEEE Transactions on Reliability* 44, 689-693.