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# Reliability and maintainability characteristics in semi-Markov models

### Keywords

semi-Markov model, reliability characteristics

### Abstract

The characteristics of semi-Markov process we can translate on the reliability characteristics in the semi-Markov reliability model. The cumulative distribution functions of the first passage time from the given states to subset of states, expected values and second moments corresponding to them which are considered in this paper allow to define reliability function of the system. The equations for many reliability characteristics and parameters are here presented.

## 1. Introduction

The semi-Markov processes were introduced independently and almost simultaneously by P. Levy [9], W.L. Smith [12] and L. Takacs [13] in 1954-55. The essential developments of semi-Markov processes theory were proposed by R. Pyke [11], [12], [13], E.Cinlar [2], Koroluk, Turbin [8], [9], N.Limnios [10], D.C. Silvestrov [14].
A semi-Markov process is constructed by the Markov renewal process which is defined by the renewal kernel and the initial distribution or by another characteristics which are equivalent to the renewal kernel. Those quantities contain full information about the process and they allow us to find many characteristics and parameters of the process, that we can translate on the reliability characteristics in the semi-Markov reliability model. The cumulative distribution functions of the first passage time from the given states to subset of states,expected values and second moments corresponding to them, enable to calculate a reliability function, a mean time to failure and another reliability parameters of an object. The equations for these quantities are here presented. Moreover,is discussed a concept of interval transition probabilities and the corresponding equations are also derived. At the end of the paper there are shownillustrative examples.

## 2. Time to failure

Assume that evolution of a system reliability is describe by a finite state space $S$semi-Markov process $\{X(t): t \geq 0\}$. Elements of a set $S$ represent the reliability states of the system. Let $S_+$ consists of the functioning states (up states) and $S_-$ contains all the failed states (down states). The subset $S_+$ and $S_-$ form a partition of $S$ , i.e., $S = S_+ \cup S_-$ and $S_+ \cap S_- = \emptyset$.
Let $\{X(t): t \geq 0\}$ be the continuous-time semi-Markov process with a discrete state space $S$ and a kernel $Q(t), t \geq 0$. A value of random variable

$$\Delta_A = \min\{n \in : X(\tau_n) \in A\}$$

denotes a discrete time (a number of state changes) of*a first arrival* at the set of states $A \subset S$ of the embedded Markov chain $\{X(\tau_n): n \in \mathrm{N}_0\}$. A value of a random variable

$$\Theta_A = \tau_{\Delta_A}(2)$$

denotes a first passage time to the subset$A$ or the time of *a first arrival* at the set of states $A \subset S$ of semi-Markov process $\{X(t): t \geq 0\}$. A function

$$\Phi_{iA}(t) = P(\Theta_A \leq t |\, X(0) = i),\ \ t \geq 0,$$

is Cumulative Distribution Function (CDF) of a random variable $\Theta_{iA}$ denoting the first passage time from the state $i \in A'$ to the subset $A$. Thus

$$\Phi_{iA}(t) = P(\Theta_{iA} \leq t).$$

If $A = S_-$ and $i \in S_+$ then $\Phi_{iA}(t), t \geq 0$, is CDF of a time to failure of an object if initial state is $i \in S_+$. For the regular semi-Markov processes such that,

$$f_{iA} = P(\Delta_A < \infty | X(0) = i) = 1,\ i \in A',$$

the distributions $\Phi_{iA}(t),\ i \in A'$ are proper and they are the only solutions of the equations system

$$\Phi_{iA}(t) = \\ = \sum_{j\in A} Q_{ij}(t) + \sum_{k\in S} \int_0^t \Phi_{kA}(t-x) dQ_{ik}(x), \\ i \in A'.$$

Applying a Laplace-Stieltjes L-S transformation for this system of integral equations we obtain the linear system of equations for (L-S) transforms

$$\tilde{\varphi}_{iA}(s) = \sum_{j\in A} \tilde{q}_{ij}(s) + \sum_{k\in A'} \tilde{q}_{ik}(s)\tilde{\varphi}_{kA}(s),$$

where

$$\tilde{\varphi}_{iA}(s) = \int_0^\infty e^{-st} d\Phi_{iA}(t),$$

are L-S transforms of the unknown CDF of the random variables $\Theta_{iA}, i \in A'$ and

$$\tilde{q}_{ij}(s) = \int_0^\infty e^{-st} dQ_{ij}(t).$$

are L-S transforms of the given functions $Q_{ij}(t), i,j \in S$. That linear system of equations is equivalent to the matrix equation

$$(I - q_{A'}(s))\varphi_{A'}(s) = b(s),$$

where

$$I = [\delta_{ij}: i,j \in A']$$

is the unit matrix,

$$q_{A'}(s) = [\tilde{q}_{ij}(s): i,j \in A']$$

is the square sub-matrix of the L-S transforms of the matrix $q(s)$, while

$$\varphi_{A'}(s) = [\tilde{\varphi}_{iA}(s): i \in A']^T, \\ b(s) = [\sum_{j\in A} \tilde{q}_{ij}(s): i \in A']^T$$

are one column matrices of the corresponding L-S transforms. The linear system of equations for the L-S transforms allows us to obtain the linear system of equations for the moments of random variables $\Theta_{iA}, i \in A'$. The expectations $E(\Theta_{iA}), i \in A'$ we obtain by solving the equation

$$(I - P_{A'})\overline{\Theta}_{A'} = \overline{T}_{A'},$$

where

$$P_{A'} = [p_{ij}: i,j \in A'],\ \overline{\Theta}_{A'} = [E(\Theta_{iA}): i \in A']^T, \\ \overline{T}_{A'} = [E(T_i): i \in A'] \quad (4)$$

and $I$ is the unit matrix. To find the second moments $E(\Theta_{iA}), i \in A'$ we have to solve the matrix equation

$$(I - P_{A'})\overline{\Theta}^2_{A'} = B_A,$$

where

$$P_{A'} = [p_{ij}: i,j \in A'],\ \overline{\Theta}_{A'} = [E(\Theta^2_{iA}): i \in A']^T, \\ B_A = [b_{iA}: i \in A']^T, \\ b_{iA} = E(T_i^2) + 2\sum_{k\in A'} p_{ik} E(T_{ik}) E(\Theta_{kA}). \quad (5)$$

## 3. Interval transition probabilities

The conditional probability (6)

$$P_{ij}(t) = P(X(t) = i\, | X(0) = i), \\ i,j \in S, t \geq 0$$

is called *the interval transition probability from the state* $i$ *to the state* $j$ in the interval $[0,t]$. The number $P_{ij}(t)$ denotes the probability that the SM process $\{X(t): t \geq 0\}$ will occupy the state $j$ at an instant $t$ if it starts from the state $i$ at moment $0$. (7)
Interval transition probabilitie satisfy the system of equations (8)

$$P_{ij}(t) = \delta_{ij}[1 - G_i(t)] + \\ + \sum_{k\in S} \int_0^t P_{kj}(t-x) dQ_{ik}(x), \qquad i,j \in S \quad (9)$$

We can obtain the solution of that system of equations applying Laplace transformation. Let

$$\tilde{P}_{ij}(s) = \int_0^\infty e^{-st} P_{ij}(t) dt, \\ \tilde{q}_{ik}(s) = \int_0^\infty e^{-st} dQ_{ik}(t), \\ \tilde{G}_i(s) = \int_0^\infty e^{-st} G_i(t) dt. \quad (10)$$

Passing to Laplace transform we obtain

$$\tilde{P}_{ij}(s) = \delta_{ij}\left[\frac{1}{s} - \tilde{G}_i(s)\right] + \sum_{k\in S} \tilde{q}_{ik}(s)\tilde{P}_{kj}(s), \\ i,j \in S.$$

If we place this system of equations in matrix form we get

$$\tilde{P}(s) = \left(\frac{1}{s}I - \tilde{G}(s)\right) + \tilde{q}(s)\tilde{P}(s),$$

where

$$\tilde{P}(s) = [\tilde{P}_{ij}(s): i,j \in S],$$
$$\tilde{G}(s) = [\delta_{ij}\tilde{G}_i(s): i,j \in S],$$
$$\ddot{q}(s) = [\tilde{q}_{ij}(s): i,j \in S],$$
$$I = [\delta_{ij}: i,j \in S].$$

## 4. The limiting probabilities

In many cases the interval transitions probabilities $P_{ij}(t), t \geq 0$ and the states probabilities

$$P_j(t) = P(X(t) = j),\ t \geq 0, j \in S$$

approach the constant values for large $t$. Let

$$P_j = \lim_{t\to\infty} P_j(t) = \lim_{t\to\infty} P(X(t) = j),\ j \in S,$$
$$P_{ij} = \lim_{t\to\infty} P_{ij}(t) = $$
$$= \lim_{t\to\infty} P(X(t) = j|X(0) = i),\ i,j \in S.$$

As a conclusion from theorems presented by Korolyuk and Turbin [7], we introduce the following theorem [5], [6]. Let $\{X(t): t \geq 0\}$ be the regular semi-Markov process with the discrete state $S$ and the kernel $Q(t) = [\ Q_{ij}(t): i,j \in S\ ]$. If the embedded Markov chain $\{X(\tau_n): n \in N_0\}$ of the semi-Markov process $\{X(t): t \in R_+\}$ contains one positive recurrent class $C$, such that for

$$\forall_{i\in S, j\in C}\ f_{ij} = 1,$$
$$\exists_{a>0}\ \forall_{i\in S}\ 0 < E(T_i) = \int_0^\infty [1 - G_i(t)]dt,$$

then there exist the limiting probabilities of $P_{ij}(t)$, $i,j \in S$ and $P_j(t)$, $j \in S$ as $t \to \infty$. Moreover

$$P_{ij} = \lim_{t\to\infty} P_{ij}(t) = P_j = \lim_{t\to\infty} P_j(t)$$
$$= \frac{\pi_j E(T_j)}{\sum_{k\in S} \pi_k E(T_k)},$$

where $\pi_i, i \in S$ form the stationary distribution of the embedded Markov chain $\{X(\tau_n): n \in N_0\}$.
Let us recall, that the stationary distribution of $\{X(\tau_n): n \in N_0\}$ is the unique solution of the linear system of equations

$$\sum_{i\in S} \pi_i p_{ij} = \pi_j, j \in S,\ \sum_{j\in S} \pi_j = 1,$$

where

$$p_{ij} = \lim_{t\to\infty} Q_{ij}(t). \tag{15}$$

## 5. Reliability and maintainability characteristics

Suppose that $i \in S_+$ is an initial state of the process. The conditional reliability function is defined by

$$R_i(t) = P(\forall u \in [0,t], X(u) \in S_+ | X(0) = i),$$
$$i \in S_+.$$

Note that from the Chapman-Kolmogorov property of a two dimensional Markov chain $\{X(\tau_n), \tau_n: n\}$, we obtain

$$R_i(t) = 1 - G_i(t) + \sum_{j\in S+} \int_0^t R_j(t-u)dQ_{ij}(u), i \in S_+. \tag{16}$$

Passing to the Laplace transform we get

$$\tilde{R}_i(s) = \frac{1}{s} - \tilde{G}_i(s) + \sum_{j\in S_+} \tilde{q}_{ij}(s)\tilde{R}_j(s), \tag{17}$$
$$i \in S_+, \tag{18}$$

where

$$\tilde{R}_j(s) = \int_0^\infty e^{-st} R_j(t)dt.$$

The matrix form of the equation system is

$$\left(I - q_{S_+}(s)\right)R(s) = W_{S_+}(s),$$

where

$$R(s) = [\tilde{R}_i(s): i \in S_+]^T,$$
$$W_{S_+}(s) = [\tfrac{1}{s} - \tilde{G}_i(s): i \in S_+]^T$$

are one column matrices, and

$$q_{S_+}(s) = [\tilde{q}_{ij}(s): i,j \in S_+],$$
$$I = [\delta_{ij}: i,j \in S_+] \tag{19}$$

are square matrices. Note that

$$\tilde{G}_i(s) = \frac{1}{s}\sum_{j\in S_+} \tilde{q}_{ij}(s).$$

Elements of the matrix $\tilde{R}(s)$ are the Laplace transforms of the conditional reliability functions.We obtain the reliability functions $R_i(t)$, $i \in S_+$ by inverting the Laplace transforms $\tilde{R}_i(s), i \in S_+$. (20)

If $A = S_-$, and $A' = S_+$, then the first passage time from a state $i \in A'$ to a subset $A$ denotes a time to failure or lifetime of the system under condition that $i$ is an initial state. Therefore the random variable $\Theta_{S_-}$ is time to failure of the system. Notice that

$$R_i(t) = P\left(\forall_{u\in[0,t]}\, X(u) \in S_+ \middle| X(0) = i\right) = P(\Theta_{S_-} > t|X(0) = i) = 1 - P\left(\Theta_{S_-} \le t\middle|X(0) = i\right) = 1 - \Phi_{iS_-}(t),\ \ i \in S_+.$$

A conditional mean time to failure (MTTF) is conditional expectation

$$\overline{\Theta}_{iS_-} = E\left(\Theta_{S_-}\middle|X(0) = i\right) = \int_0^\infty R_i(t)dt, i \in S_+.$$

A corresponding second moment is

$$\overline{\Theta}^2{}_{iS_-} = E(\Theta_{S_-}|X(0) = i) = 2\int_0^\infty t\, R_i(t)dt, i \in S_+.$$

We can calculate those reliability parameters of the system in another wa. We can solve equations (8) substituting $A = S_-$, and $A' = S_+$.
Unconditional reliability function of the system is defined as

$$R(t) = P(\forall u \in [0,t], X(u) \in S_+).$$

Note that

$$R(t) = P(\forall u \in [0,t], X(u) \in S_+) = \sum_{i\in S+} p_i\, R_i(t),$$

as

$$P(X(0) = i) = 0 \text{ for } i \in S_-.$$

## 6. Pointwise availability

The availability at time *t* is determined as

$$A(t) = P(X(t) \in S_+).$$

Notice that

$$A(t) = P(X(t) \in S_+) = P(X(t) \in S_+, X(0) \in S) = \sum_{i\in S}\sum_{i\in S_+} P(X(t) = j|X(0) = i) \cdot P(X(0) = i) = \sum_{i\in S}\sum_{i\in S_+} p_i\, P_{ij}(t). \quad (24)$$

From the equality it follows that we can calculate the pointwise availability having an initial distribution $p_i = P(X(0) = i), i \in S$ and the interval transition probabilities $P_{ij}(t), i \in S, j \in S_+$.
If there exist the limit probabilities

$$P_j = \lim_{t\to\infty} P_{ij}(t), j \in S_+,$$

then the limiting (steady-state) availability (25)

$$A = \lim_{t\to\infty} A(t)$$

is given by

$$A = \sum_{j\in S_+} P_j. \quad (26)$$

## 7. Maintainability function and parameters of the system

In similar way we obtain the maintainability function and parameters of the system, symmetrically to the reliability case. Now we assume that $i \in S_-$ is an initial state of the process. The conditional maintainability function is determined by

$$M_i(t) = P\left(\forall_{u\in[0,t]}, X(u) \in S_-\middle|X(0) = i\right),\ \ i \in S_-. \quad (27)$$

These functions fulfil the equations system

$$M_i(t) = 1 - G_i(t) + \sum_{j\in S_-} \int_0^t M_j(t-u)dQ_{ij}(u),\ \ i \in S_-, \quad (28)$$

which is equivalent to the system of equations for the Laplace transform

$$\widetilde{M}_i(s) = \frac{1}{s} - \tilde{G}_i(s) + \sum_{j\in S_-} \tilde{q}_{ij}(s)\widetilde{M}_j(s),\ \ i \in S_-,$$

where

$$\widetilde{M}_j(s) = \int_0^\infty e^{-st} M_j(t)dt. \quad (29)$$

We obtain the maintainability functions $M_i(t),\ i \in S_-$ by inverting the Laplace transforms $\widetilde{M}_i(s),\ i \in S_-$ being a solution of the above system of equations.

If $A = S_+$, and $A' = S_-$, then the first passage time from a state $i \in A'$ to a subset $A$ denotes a *time to repair* or a *maintenance time* of the system under condition that $i \in S_-$ is an initial state. Therefore the random variable $\Theta_{S_+}$ denotes the maintenance time of the system. Notice that

$$\begin{aligned} M_i(t) &= P(\forall u \in [0,t], X(u) \in S_- | X(0) = i) \\ &= P(\Theta_{S_+} > t | X(0) = i) \\ &= 1 - P\big(\Theta_{S_-} \le t \big| X(0) = i\big) \\ &= 1 - \Phi_{iS_+}(t) \;\; i \in S_-. \end{aligned} \tag{34}$$

The conditional mean maintenance time (MMT) and the corresponding second moment are

$$\begin{aligned} \overline{\Theta}_{iS_+} &= E(\Theta_{S_+} | X(0) = i) \\ &= \int_0^\infty M_i(t)dt, i \in S_-. \end{aligned} \tag{35}$$

$$\begin{aligned} \overline{\Theta}^2{}_{iS_+} &= E(\Theta_{S_+}^2 | X(0) = i) \\ &= 2\int_0^\infty t\, M_i(t)dt, i \in S_-. \end{aligned} \tag{36}$$

We can also calculate those parameters of the system in another way. We can solve equations (11) and (12), substituting $A = S_+$, and $A' = S_-$.
The unconditional maintainability function of the system is defined as

$$M(t) = P(\forall u \in [0,t], X(u) \in S_-). \tag{37}$$

Note that

$$\begin{aligned} M(t) &= P(\forall u \in [0,t], X(u) \in S_-) \\ &= \sum_{i \in S-} p_i\, M_i(t). \end{aligned} \tag{38}$$

## 8. Illustrative example

### 8.1. Description and assumptions

The object (device) works by performing the two types of tasks 1 and 2. Duration of task $k$ is a nonnegative random variable $\xi_k, k = 1{,}2$ governed by a CDF $F_{\xi_k}(x), x \ge 0, k = 1{,}2$. Working object may be damaged. Time to failure of the object executing a task $k$ is a nonnegative random variable $\zeta_k, k = 1{,}2$ with probability density function $f_{\zeta_k}(x)$, $x \ge 0$, $k = 1{,}2$. A repair (renewal, maintenance) time of the object performing task $k$ is a nonnegative random variable $\eta_k, k = 1{,}2$ governed by probability density function $f_{\eta_k}(x), x \ge 0, k = 1{,}2$. After each operation or maintenance the object waits for the next task. In this case a waiting time is a nonnegative random variable $\gamma$, having PDF $f_\gamma(x), x \ge 0$, $k = 1{,}2$ where in it is task 1 with a probability $p$ or task 2 with a probability $q = 1 - p$. Furthermore we assume that all random variables and their copies are independent and they have the finite second moments.

### 8.2. Model construction

We start the construction of the model with the determination of the operation process states:
1 - state of waiting for the tasks performing
2 - performing of the task 1
3 - performing of the task 2
4 - repair after failure during executing of the task 1
5 - repair after failure during executing of the task 2.

A model of a object operation is a semi-Markov process with a state space $S = \{1,2,3,4,5\}$ and the kernel

$$Q(t) = \begin{bmatrix} 0 & Q_{12}(t) & Q_{13}(t) & 0 & 0 \\ Q_{21}(t) & 0 & 0 & Q_{24}(t) & 0 \\ Q_{31}(t) & & 0 & 0 & Q_{35}(t) \\ Q_{41}(t) & 0 & 0 & 0 & 0 \\ Q_{51}(t) & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{39}$$

The model is constructed if all kernel elements are determined. According to assumptions we calculate elements of the matrix $Q(t)$. In detail we explain calculation of an element $Q_{21}(t)$. Transition from a state 2 to 1 in time no grater than $t$ takes place if duration of task 1, denoting as $\xi_1$, is less or equal $t$ and time to failure is greater then $\xi_1$. Hence

$$\begin{aligned} Q_{21}(t) &= P(\xi_1 \le t, \zeta_1 > \xi_1) \\ &= \iint_{D_{21}} dF_{\xi_1}(x) dF_{\zeta_1}(y), \end{aligned}$$

where

$$D_{21} = \{(x,y): 0 \le x \le t, y > x\}.$$

Therefore

$$\begin{aligned} Q_{21}(t) &= \int_0^t dF_{\xi_1}(x) \int_x^\infty dF_{\zeta_1}(x) \\ &= \int_0^t [1 - F_{\zeta_1}(x)] dF_{\xi_1}(x), \end{aligned}$$

where

$$F_{\zeta_1}(x) = \int_0^t f_{\zeta_1}(x)dx.$$

In similar way we calculate the rest of elements. Finally we obtain:

$$Q_{12}(t) = p\, F_\gamma(t),$$
$$Q_{13}(t) = q\, F_\gamma(t),$$
$$Q_{21}(t) = \int_0^t [1 - F_{\zeta_1}(x)]dF_{\xi_1}(x)$$
$$Q_{24}(t) = \int_0^t [1 - F_{\xi_1}(x)]dF_{\zeta_1}(x)$$
$$Q_{31}(t) = \int_0^t [1 - F_{\zeta_2}(x)]dF_{\xi_2}(x),$$
$$Q_{35}(t) = \int_0^t [1 - F_{\xi_2}(x)]dF_{\zeta_2}(x),$$
$$Q_{41}(t) = F_{\eta_1}(t),$$
$$Q_{51}(t) = F_{\eta_2}(t).$$

The transition probability matrix of the embedded Markov chain $\{X(\tau_n): n \in N_0\}$ is of the form:

$$P = \begin{bmatrix} 0 & p & q & 0 & 0 \\ p_{21} & 0 & 0 & p_{24} & 0 \\ p_{31} & 0 & 0 & 0 & p_{35} \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$p_{21} = \int_0^\infty [1 - F_{\zeta_1}(x)]dx\,,$$
$$p_{24} = \int_0^\infty [1 - F_{\xi_1}(x)]dF_{\zeta_1}(x) = 1 - p_{21},$$
$$p_{31} = \int_0^\infty [1 - F_{\zeta_2}(x)]dF_{\xi_2}(x),$$
$$p_{35} = \int_0^\infty [1 - F_{\xi_2}(x)]dF_{\zeta_2}(x) = 1 - p_{31}.$$

The Laplace-Stieltjes transform of the kernel (39) is given by

$$\tilde{q}(s) = \begin{bmatrix} 0 & \tilde{q}_{12}(s) & \tilde{q}_{13}(s) & 0 & 0 \\ \tilde{q}_{21}(s) & 0 & 0 & \tilde{q}_{24}(s) & 0 \\ \tilde{q}_{31}(s) & & 0 & 0 & \tilde{q}_{35}(s) \\ \tilde{q}_{41}(s) & 0 & 0 & 0 & 0 \\ \tilde{q}_{51}(s) & 0 & 0 & 0 & 0 \end{bmatrix}.$$

## 8.3. Reliability characteristics and parameters

In the model a subset $S_+ = \{1,2,3\}$ consists of the functioning states and a subset $S_- = \{4,5\}$ contains all failed states of the object operation process. In this case the matrix equation (23) for the Laplace transform of the conditional reliability function takes the form

$$\begin{bmatrix} 1 & -\tilde{q}_{12}(s) & -\tilde{q}_{13}(s) \\ -\tilde{q}_{21}(s) & 1 & 0 \\ -\tilde{q}_{31}(s) & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{R}_1(s) \\ \tilde{R}_2(s) \\ \tilde{R}_3(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{s}(1 - \tilde{q}_{12}(s) - \tilde{q}_{13}(s)) \\ \frac{1}{s}(1 - \tilde{q}_{21}(s) - \tilde{q}_{24}(s)) \\ \frac{1}{s}(1 - \tilde{q}_{31}(s) - \tilde{q}_{35}(s)) \end{bmatrix}.$$

We obtain the reliability functions $R_i(t), i \in S_+$ by inverting the Laplace transforms $\tilde{R}_i(s), i \in S_+$.
The numbers $\overline{\Theta}_{iS_-} = E(\Theta_{iS_-}),\ i \in S_+$ signify conditional mean times to failure. They are unique solutions of the linear equations system (11), which in this case have the following matrix form

$$(I - P_{S_+})\overline{\Theta}_{S_+} = \overline{T}_{S_+},$$
$$P_{S_+} = [p_{ij}: i, j \in S_+],$$
$$\overline{\Theta}_{S_+} = [E(\Theta_{iS_-}): i \in S_+]^T,$$
$$\overline{T}_{S_+} = [E(T_i): i \in S_+]^T \qquad (40)$$

and $I$ is the unit matrix. Hence

$$\begin{bmatrix} 1 & -p & -\tilde{q} \\ -p_{21} & 1 & 0 \\ -p_{31} & 0 & 1 \end{bmatrix} \begin{bmatrix} E(\Theta_{1S_-}) \\ E(\Theta_{2S_-}) \\ E(\Theta_{3S_-}) \end{bmatrix} = \begin{bmatrix} E(T_1) \\ E(T_2) \\ E(T_3) \end{bmatrix},$$

$$\overline{g}_1 = E(T_1) = E(\gamma),$$
$$\overline{g}_2 = E(T_2) = E(\min\{\xi_1, \zeta_1\}),$$
$$\overline{g}_3 = E(T_3) = E(\min\{\xi_2, \zeta_2\}).$$

Solving the equation we get

$$E(\Theta_{1S_-}) = \frac{\overline{g}_1 + p\overline{g}_2 + q\overline{g}_3}{1 - p\, p_{21} - q\, p_{31}},$$
$$E(\Theta_{2S_-}) = \frac{p_{21}\overline{g}_1 + (1 - q\, p_{31})\overline{g}_2 + q\, p_{21}\overline{g}_3}{1 - p\, p_{21} - q\, p_{31}},$$
$$E(\Theta_{3S_-}) = \frac{p_{31}\overline{g}_1 + p\, p_{31}\overline{g}_2 + (1 - p\, p_{21})\overline{g}_3}{1 - p\, p_{21} - q\, p_{31}}. \qquad (41)$$

To obtain the limiting distribution of the process states, the steady-state availability, the conditional means time to failure and the corresponding second moments we have to solve the appropriate system of linear equations.

## References

[1] Barlow, RE. & Proshan, F. (1975). *Statistical theory of reliability and life testing*. New York: Holt, Rinchart and Winston, Inc.

[2] Cinlar, E. (1969). Markov renewal theory. *Adv.*

*Appl. Probab* 2, 1, 123-187.

[3] Feller, W. (1964). On semi-Markov processes. *Proc. Nat. Acad. Sci*. USA. 51, 4, 653-659.

[4] Grabski, F. (1982). *Theory of Semi-Markov Operation Processes*. ZN AMW, Gdynia. 75A. (in Polish).

[5] Grabski, F. (2002). *Semi-Markov models of reliability and operation*. IBS PAN. Warsaw. (in Polish).

[6] Grabski, F. (2014). *Semi-Markov Processes: Applications in Systems Reliability and Maintenance*. Elsevier; Amsterdam, Boston, Heidelberg, London, New York, Oxford, Paris, San Diego, San Francisco, Sydney.

[7] Korolyuk, VS. & Turbin, A. F. (1976). *Semi-Markov processes and their applications*. Naukova Dumka, Kiev. (in Russian).

[8] Korolyuk, V. S. & Turbin A. F. (1982). *Markov Renewal Processes in Problems of Systems Reliability*. Naukova Dumka, Kiev. (in Russian).

[9] Lev′y, P. (1954). Proceesus semi-markoviens. *Proc. Int. Cong. Math*. Amsterdam, 416-426.

[10] Limnios, N. & Oprisan, G. (2001). *Semi-Markov Processes and Reliability*. Birkhauser, Boston.

[11] Pyke, R. (1961). Markov renewal processes: definitions and preliminary properties. *Ann. of Math. Statist*. 32, 1231-1242.

[12] Pyke, R. (1961). Markov renewal processes with finitely many states. *Ann. of Math. Statist.* 32, 1243-1259.

[13] Pyke, R. & Schaufele, R. (1964). Limit theorems for Markov renewal processes. *Ann. of Math. Statist* 32, 1746-1764.

[14] Silvestrov, DC. (1980). *Semi-Markov processes with a discrete state space*. Sovetskoe Radio, Moskow. (in Russian).

[15] Smith, WL. (1955). Regenerative stochastic processes. *Proc.Roy.Soc*. London. A, 232, 6-31.

[16] Takács, L. (1954). Some investigations concerning recurrent stochastic processes of a certain type. *Magyar Tud. Akad. Mat. Kutato Int. Kzl* 3, 115-128.