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A Degradation-Threshold-Shock model for a system. The case of dependent causes of failure in finite-time

Keywords

Condition-based maintenance, degradation-threshold-shock models, finite-time analysis, gamma process

Abstract

This paper deals with a condition-based maintenance strategy (CBM) in finite-time horizon for a system subject to two different causes of failure, internal degradation and sudden shocks. Internal degradation is modelled under a gamma process and sudden shocks arrive at the system following a non-homogeneous Poisson process (NHPP). Both causes of failure are considered as dependent. When a sudden shock takes places, the system fails. In addition, the system is regarded to fail when the deterioration level reaches a critical threshold. Under this functioning scheme, a CBM strategy is developed for controlling the reliability of the system. Traditionally, this strategy is developed under an asymptotic approach. Hence, considering an infinite-time horizon is not always possible. In this paper, we analyse a CBM strategy under a finite-time horizon developing a recursive method which estimates the expected cost rate based on numerical integration and Monte Carlo simulation. A numerical example is provided in order to illustrate this complex maintenance model.

1. Introduction

Most systems are degraded over time. This degradation process complicates maintenance tasks since it is uncertain and depends on the time. Maintenance strategies regulate maintenance tasks which must be performed on the system. Establishing a good maintenance task, we can ensure the correct functioning of the system and optimize the maintenance cost.

The gamma process is a stochastic cumulative process, which is considered as one of the most appropriated processes for modeling the damage involved by the cumulative deterioration of systems and structures [9].

However, systems are not only subject to internal degradation, but also are exposed to sudden shocks which can cause its failure. Sudden shocks arrive at the system following a non-homogeneous Poisson process (NHPP). Up to we know [7] were the first to combine both causes of failure, proposing Degradation-Threshold-Shock (DTS) models. Thus, the failure time of a system subject to these two causes of failure is the minimum instant between when degradation reaches a critical threshold and when a sudden shock occurs.

Traditionally, condition-based maintenance (CBM) strategies have been programmed for controlling the reliability in DTS models. In most papers, the criterion for optimizing CBM strategies is the asymptotic cost rate [2]-[3], [5]-[6] which equals the expected cost in a renewal cycle divided by the length of the renewal cycle. However, most systems have a finite operating life-time due to, when a system fails; it cannot always be replaced by a new one with the same characteristics as the previous one. In those cases, the use of the asymptotic approach is questionable and the cost must be analyzed in a transient way.

Although maintenance policies which consider a finite-time horizon are more realistic than those considering an infinite-time horizon, the first ones are less used due to the analytical and computational difficulty of treatment that they involve.

[4] and [8] proposed CBM strategies in finite-time for systems subject to a degradation process modeled by using a gamma process where time was considered as a discrete time. This paper expands these works by considering the time as a continuous

variable and by adding a new component of risk (sudden shocks), whose arrival depends on the internal degradation process of the system. Up to our knowledge, a CBM strategy in a DTS model in a finite-time horizon for a system has not been considered yet.

In this paper, we propose a CBM strategy for a system subject to two dependent causes of failure: internal degradation modeled under a gamma process and sudden shocks which follow an NHPP. Both causes of failure are dependent in the sense that the system is more susceptible to a sudden shock when the deterioration level of the system reaches a certain threshold. Under this framework, a recursive method which combines numerical integration and Monte Carlo simulation is proposed in order to obtain the expected cost in finite-time horizon. A numerical example is provided for illustrating this complex maintenance model.

2. Framework of the problem

The general assumptions of this model are:

- The system starts working at time $t = 0$. This system is subject to an internal degradation process which evolves depending on the environmental conditions and the components according to a gamma process with parameters α and β for $\alpha, \beta > 0$. Let $X(t)$ be the deterioration level of the system at time $t = 0$. Thus, for two time instants s and t, with $s < t$, the density function of the increment deterioration $X(t) - X(s)$ is given by

  $$f_{\alpha(t-s),\beta}(x) = \frac{\beta^{\alpha(t-s)}}{\Gamma(\alpha(t-s))} x^{\alpha(t-s)-1} e^{-\beta x} \quad (1)$$

  for $x > 0$, where $\Gamma(\cdot)$ denotes the gamma function defined as

  $$\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u}\, du. \quad (2)$$

  The system fails due to degradation when the deterioration level exceeds a known and fixed threshold L, named breakdown threshold.

- The system not only fails due to internal degradation, but also this is subject to sudden shocks which can cause its failure. Sudden shocks arrive at the system according to the NHPP $\{N_s(t), t > 0\}$. This process shows the dependence between both causes of failure, internal degradation and sudden shocks since there is an interaction between both density function corresponding to each cause of failure. That means the intensity of the sudden shocks processes at time t depends on the deterioration level of the system. This dependence is reflected in the fact that the system is more susceptible to a sudden shock when the deterioration level of the system reaches a certain fixed threshold M_s. Then, $\{N_s(t), t > 0\}$ is an NHPP with intensity

  $$\lambda(t) = \lambda_1(t)\mathbf{1}_{\{X(t)\leq M_s\}} + \lambda_2(t)\mathbf{1}_{\{X(t)>M_s\}}, t \geq 0,$$

  where λ_1 and λ_2 denote two failure rate functions which verify $\lambda_1 \leq \lambda_2$, for all $t \geq 0$, where t is the age of the system and $\mathbf{1}_{\{\cdot\}}$ denote the indicator function which equals 1 if the argument is true and 0 otherwise. A sudden shock provokes the total failure of the system.

- The system is inspected each T $(T > 0)$ time units $(t.u.)$ with the purpose of checking if the system is working or is down. If the system is down, a corrective maintenance (CM) is performed. A simulation of a CM event is shown in *Figure 1*.

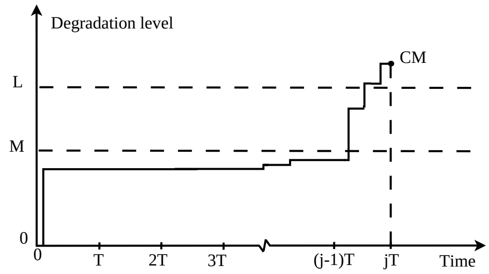

Figure 1. A corrective maintenance event

On the other hand, if the system is still working, its deterioration level is checked. Let M be for $M < L$, the deterioration level from which the system is considered as too worn as to be replaced in a preventive way. If the system is still working and the deterioration level of the system is reached the preventive threshold M, a preventive maintenance (PM) is performed and A simulation of a PM event is shown in *Figure 2*.

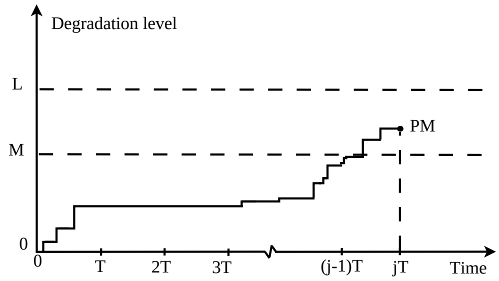

Figure 2. A preventive maintenance event

Both maintenance tasks, corrective and preventive maintenance, imply the replacement of the system by a new one with identical conditions to the previous one. Otherwise no maintenance action is performed. We suppose that the time required to perform a maintenance action is negligible.

- Each maintenance action implies a cost. A CM and a PM have associated a cost of C_c and C_p monetary units $(m.u.)$, respectively. An inspection implies a cost of $C_i\ m.u.$ In addition, if the system fails, the system is down until the next inspection. Each time unit that the system is down, a cost $C_d\ m.u./t.u.$ is incurred. We assume that $C_c > C_p > C_i$.
- Periodic and sequential inspections policies and maintenance tasks are performed until a finite time $t_f > 0$, when finite operating life-time of the system finishes and the system cannot be replaced by a new one with the same characteristics as the previous one.

Let σ_z be the random variable which is the time to a certain level z is reached. Denoting F_{σ_z} the distribution function of σ_z, we have

$$F_{\sigma_z}(t) = P[\sigma_z \leq t] = P[X(t) \geq z] = \int_z^{\infty} f_{\alpha t,\beta}(x)\, dx = \frac{\Gamma(\alpha t, z\beta)}{\Gamma(\alpha t)}, \qquad (3)$$

for $t \geq 0$ where $f_{\alpha t,\beta}(x)$ and $\Gamma(\alpha t)$ are given by (1) and (2), respectively, and

$$\Gamma(\alpha, x) = \int_x^{\infty} u^{\alpha-1} e^{-u}\, du,$$

denotes the incomplete gamma function for $x \geq 0$ and $\alpha > 0$.

Let σ_L, σ_M, and σ_{M_s} be the random variables which denote the first time where the degradation levels L, M, and M_s are reached, respectively, and let F_{σ_L}, F_{σ_M}, and $F_{\sigma_{M_s}}$ be the distribution functions of the variables σ_L, σ_M, and σ_{M_s} given by (3), respectively.

The probability of the random variable $\sigma_{z_2} - \sigma_{z_1}$ for $z_1 \leq z_2$ is provided as

$$\bar{F}_{\sigma_{z_2}-\sigma_{z_1}}(t) = P[\sigma_{z_2} - \sigma_{z_1} \geq t] = \int_{x=0}^{\infty}\int_{y=z_1}^{\infty} f_{\sigma_{z_1},X(\sigma_{z_1})}(x,y)\, F_{\alpha t,\beta}(z_2 - y)\, dy\, dx,$$

where $F_{\alpha t,\beta}$ denotes the distribution function of the density function $f_{\alpha t,\beta}$ given by (1) and $f_{\sigma_{z_1},X(\sigma_{z_1})}$ is the joint density function of $(\sigma_{z_1}, X(\sigma_{z_1}))$ provided by Bertoin [1] as

$$f_{\sigma_{z_1},X(\sigma_{z_1})}(x,y) = \int_0^{\infty} \mathbf{1}_{\{z_1 \leq y < z_1+s\}}\, f_{\alpha t,\beta}(y-s)\mu(ds),$$

where $\mu(ds)$ denotes the Lévy measure associated to a gamma process with parameters α and β given by

$$\mu(ds) = \alpha\frac{e^{-\beta s}}{s}, \qquad s > 0.$$

The system is also subject to sudden shocks which can cause the system failure. Sudden shocks occur randomly in time under an arrival rate which depends on the deterioration level of the system. A sudden shock provokes the total failure of the system. Let Y be the random variable which determines the time to the first sudden shock. Let $\sigma_{M_s} = v$ be the time where the deterioration threshold M_s. In that way, the failure rate of Y is expressed as

$$\lambda(t) = \lambda_1(t)\mathbf{1}_{\{X(t)\leq M_s\}} + \lambda_2(t)\mathbf{1}_{\{X(t)>M_s\}}, t \geq 0.$$

Let $I(v,t)$ be the survival function of Y for $t \geq 0$, knowing that the threshold M_s is reached at $\sigma_{M_s} = v$. That is

$$I(v,t) = P[Y > t|_{\sigma_{M_s}=v}] = \exp\left\{-\int_0^t \lambda(z)dz\right\} = \frac{\bar{F}_1(v)}{\bar{F}_1(0)}\frac{\bar{F}_2(t)}{\bar{F}_2(v)},$$

for $v \leq t$, where

$$F_j(t) = \exp\left\{-\int_0^t \lambda_j(z)dz\right\},$$

with density function $f_j(t)$ for $j = 1,2$.

3. Expected cost analysis in finite-time

Let $C(t)$ be the maintenance cost of the system at time $t > 0$. In most papers of the current literature, the optimization criterion in a CBM for a system is the asymptotic cost rate (asymptotic cost per time unit).

Let C^{∞} be the asymptotic cost rate. Based on the *"Renewal Theorem"*, C^{∞} equals the expected cost in a renewal cycle divided by the length of the renewal cycle. That is

$$C^{\infty} = \frac{\mathrm{E}[\mathrm{C}_1]}{\mathrm{E}[\mathrm{R}_1]},$$

being C_1 and R_1 the cost and the length of the first renewal cycle, respectively.

However, most systems have a finite operating life-time since the system cannot always be replaced by a new one with the same characteristics as the previous one. In that cases, the use of the asymptotic approach is questionable and we must analyze the maintenance cost in a transient way.

Let R_j be the length of the j-th renewal cycle, with $j = 1,2,\dots,N(t_f)$, being $N(t_f)$ the total number of complete renewal cycles to t_f. Let C_j be the associated cost to the j-th renewal cycle, for $j = 1,2,\dots,N(t_f)$, and let D_j be the chronological time of the j-th renewal, for $j = 1,2,\dots,N(t_f)$. That is

$$D_j = \sum_{n=1}^{j} R_n.$$

Thus, the length of the j-th renewal cycle R_j is given by

$$R_j = D_j - D_{j-1}, \qquad j = 1,2,\dots,N(t_f),$$

for $D_0 = 0$. A realization of this process is shown in *Figure 3*.

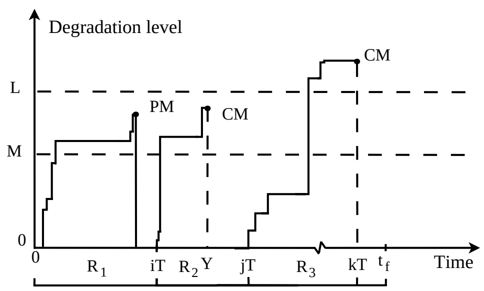

Figure 3. A realization of a renewal cycle sequence

The total cost to t_f is the sum of the incurred costs in the different $N(t_f)$ complete renewal cycles and the incurred cost of the incomplete cycle in the interval time $\left(D_{N(t_f)}, t_f\right]$. That is

$$C(t_f) = \sum_{j=1}^{N(t_f)} C_j + C\left(D_{N(t_f)}, t_f\right),$$

where $C(t_1,t_2)$ denotes the cost in the interval $(t_1,t_2]$, and $C(0,t_f)$ is expressed in a compact way as $C(t_f)$.

Let $R_1^M(kT)$ be the time to the first maintenance action, and let $R_{1,p}^M(kT)$ and $R_{1,c}^M(kT)$ be the time to the first preventive and corrective maintenance action, respectively. For a fixed time between inspections T and, let $P_{R_1}^M(kT)$, $P_{R_{1,p}}^M(kT)$, and $P_{R_{1,c}}^M(kT)$ be the following probabilities

$$P_{R_1}^M(kT) = P[R_1^M = kT],$$

$$P_{R_{1,p}}^M(kT) = P\left[R_{1,p}^M = kT\right],$$

$$P_{R_{1,c}}^M(kT) = P\left[R_{1,c}^M = kT\right],$$

for $k = 1,2,\dots,\lfloor t_f/T \rfloor$. Analytically, these probabilities are given by

$$\begin{aligned}P_{R_{1,p}}^M(kT) = \big(&\mathrm{P}[(k-1)T < \sigma_M < \sigma_{M_s} < kT < \sigma_L, Y > kT] \\ &+ \mathrm{P}[(k-1)T < \sigma_M < kT < \sigma_{M_s}, Y > kT]\big)\mathbf{1}_{\{M \le M_s\}} \\ &+ \big(\mathrm{P}[(k-1)T < \sigma_{M_s} < \sigma_M < kT < \sigma_L, Y > kT] \\ &+ \mathrm{P}[\sigma_{M_s} < (k-1)T < \sigma_M < kT < \sigma_L, Y > kT]\big)\mathbf{1}_{\{M > M_s\}},\end{aligned} \tag{4}$$

$$\begin{aligned}P_{R_{1,c}}^M(kT) = \big(&\mathrm{P}[(k-1)T < \sigma_M < \sigma_{M_s} < \sigma_L < kT, \sigma_L < Y] \\ &+\mathrm{P}[(k-1)T < \sigma_M < Y < kT, Y < \sigma_{M_s}] \\ &+\mathrm{P}[(k-1)T < \sigma_M < \sigma_{M_s} < Y < kT, Y < \sigma_L] \\ &+\mathrm{P}[(k-1)T < Y < kT, Y < \sigma_M]\big)\mathbf{1}_{\{M \le M_s\}} \\ &+\big(\mathrm{P}[(k-1)T < \sigma_{M_s} < \sigma_L < kT, \sigma_L < Y] \\ &+\mathrm{P}[\sigma_{M_s} < (k-1)T < \sigma_M < \sigma_L < kT, \sigma_L < Y] \\ &+\mathrm{P}[\sigma_{M_s} < (k-1)T < \sigma_M < Y < kT, Y < \sigma_L] \\ &+\mathrm{P}[\sigma_{M_s} < (k-1)T < Y < kT, Y < \sigma_M] \\ &+\mathrm{P}[(k-1)T < \sigma_{M_s} < Y < kT, Y < \sigma_L] \\ &+\mathrm{P}[(k-1)T < Y < kT, Y < \sigma_{M_s}]\big)\mathbf{1}_{\{M > M_s\}},\end{aligned} \tag{5}$$

and

$$P_{R_1}^M(kT) = P_{R_{1,p}}^M(kT) + P_{R_{1,c}}^M(kT). \tag{6}$$

Let $W_{T,k}^M$ be the time the system is down in the interval $((k-1)T, kT]$. Then,

$$\begin{aligned}E\left[W_{T,k}^M\right] = \big(&\mathrm{P}[(k-1)T < \sigma_M < \sigma_{M_s} < \sigma_L < kT, \sigma_L < Y](kT - \sigma_L) \\ &+\mathrm{P}\left[(k-1)T < \sigma_M < Y < kT, Y < \sigma_{M_s}\right](kT - Y) \\ &+\mathrm{P}\left[(k-1)T < \sigma_M < \sigma_{M_s} < Y < kT, Y < \sigma_L\right](kT - Y) \\ &+\mathrm{P}[(k-1)T < Y < kT, Y < \sigma_M](kT - Y)\big)\mathbf{1}_{\{M \le M_s\}} \\ &+\big(\mathrm{P}\left[(k-1)T < \sigma_{M_s} < \sigma_L < kT, \sigma_L < Y\right](kT - \sigma_L)\end{aligned} \tag{7}$$

$+\mathrm{P}[\sigma_{M_s} < (k-1)T < \sigma_\mathrm{M} < \sigma_\mathrm{L} < kT, \sigma_\mathrm{L} < Y](kT - \sigma_L)$
$+\mathrm{P}[\sigma_{M_s} < (\mathrm{k}-1)\mathrm{T} < \sigma_\mathrm{M} < Y < kT, Y < \sigma_L]\,(kT - Y)$
$+\mathrm{P}\left[\sigma_{M_s} < (\mathrm{k}-1)\mathrm{T} < Y < kT, Y < \sigma_M\right](kT - Y)$
$+\mathrm{P}\left[(\mathrm{k}-1)\mathrm{T} < \sigma_{M_s} < Y < kT, Y < \sigma_\mathrm{L}\right](kT - Y)$
$+\mathrm{P}[(\mathrm{k}-1)\mathrm{T} < Y < kT, \backslash\, Y < \sigma_{M_s}](kT - Y))\mathbf{1}_{\{M > M_s\}}.$

Let $E[C_T^M(t)]$ be the expected cost at time $t > 0$ with a time between inspections T and a preventive threshold M. For $t < T$, we have

$$E[C_T^M(t)] = C_d \int_0^t f_{\sigma_{M_s}}(u) \int_u^t \left[-\frac{\partial}{\partial v}\left(I(u,v)\bar{F}_{\sigma_L - \sigma_{M_s}}(v-u)\right)\right](t-v)\,dv\,du + C_d \int_0^t f_1(u)\bar{F}_{\sigma_{M_s}}(u)(t-u)\,du. \quad (8)$$

For $t \geq T$, $E[C_T^M(t)]$ fulfills the following recursive equation

$$E[C_T^M(t)] = \sum_{k=1}^{\lfloor t/T \rfloor} E[C_T^M(t-kT)]\, P_{R_1}^M(kT) + G_T^M(t) \quad (9)$$

where

$$G_T^M(t) = \left(\lfloor t/T \rfloor C_i(k-1) + C_d E\left[W_{\lfloor t/T \rfloor}^M\right]\right)\left(1 - \sum_{k=1}^{\lfloor t/T \rfloor} P_{R_1}^M(kT)\right) + \sum_{k=1}^{\lfloor t/T \rfloor}\left(C_p + C_i(k-1)\right)P_{R_{1,p}}^M(kT) + \sum_{k=1}^{\lfloor t/T \rfloor}\left(C_c + C_i(k-1)\right)P_{R_{1,c}}^M(kT) + \sum_{k=1}^{\lfloor t/T \rfloor} C_d E\left[W_{T,k}^M\right] P_{R_{1,c}}^M(kT),$$

with initial conditions $E[C_T^M(0)] = 0$ and $G_T^M(0) = 0$, time between inspections T $(t.u.)$ and preventive threshold M, where $P_{R_{1,p}}^M(kT)$, $P_{R_{1,c}}^M(kT)$, and $P_{R_1}^M(kT)$ are the probabilities given by (4), (5), and (6), respectively and $E\left[W_{T,k}^M\right]$ denotes the expected downtime in $((k-1)T, kT]$ given by (7).
For fixed T and variable M,

$$E\left[C_T^{M_{opt}}(t)\right] = min\{0 \leq \mathrm{M} \leq \mathrm{L};\ E[C_T^M(t)]\}.$$

4. Illustrative example

In this section we compute an estimation of the expected cost rate in finite-time horizon by using the recursive formula given in (9). To this end, we consider a system subject to a degradation process whose growth is modeled under a gamma process with parameters $\alpha = \beta = 0.1$. In this way, being $X(t)$ the deterioration level of the system at time t, the density function of $X(t)$ is given by

$$f_{0.1\mathrm{t},0.1}(\mathrm{x}) = \frac{0.1^{0.1t}}{\Gamma(0.1\mathrm{t})} x^{0.1(t-1)} e^{-0.1x}, \qquad x > 0,$$

where $\Gamma(\cdot)$ is given by (2).
The system fails when the deterioration level of the system reaches the breakdown threshold $L = 30$. The system can also fail due to a sudden shock. Both causes of failure are dependent. This dependence is reflected in the fact that the failure rate function of the sudden shock processes depends on the degradation process. Thus, the system is more susceptible to a sudden shock when the deterioration level of the system reaches a certain threshold M_s. In this example, we assume that $M_s = 20$ and the sudden shock processes is modelled under an NHPP with intensity

$$\lambda(t) = 0.01 \cdot \mathbf{1}_{\{X(t) \leq M_s\}} + 0.1 \cdot \mathbf{1}_{\{X(t) > M_s\}}, t \geq 0.$$

Under these conditions, the expected time to the failure due to deterioration is $E[\sigma_L] = 34.0335\ t.u.$ and the expected time to the failure due to a sudden shock is $E[Y] = 28.3556\ t.u.$
In addition, we assume the cost sequence $C_c = 100\,m.u.$, $C_p = 50\,m.u.$, $C_d = 25\,m.u./t.u.$, and $C_i = 2\,m.u.$ Periodical and sequential inspection policies and maintenance tasks are performed until a finite time $t_f = 10$.
We consider a time between inspections $T = 2.5\ t.u.$ Based on (8), if $t < 2.5$, the expression

$$E[C_{2.5}^M(t)] = C_d \int_0^t f_{\sigma_{M_s}}(u) \int_u^t \left[-\frac{\partial}{\partial v}\left(I(u,v)\bar{F}_{\sigma_L - \sigma_{M_s}}(v-u)\right)\right](t-v)\,dv\,du + C_d \int_0^t f_1(u)\bar{F}_{\sigma_{M_s}}(u)(t-u)\,du.$$

is estimated by using Monte Carlo simulation with 10000 realizations in MATLAB software.

If $t \geq 2.5$, the value of $E[C_{2.5}^{M}(t)]$ given by Equation (9) involves the calculus of $P_{R_{1,p}}^{M}(2.5k)$, $P_{R_{1,c}}^{M}(2.5k)$, $P_{R_1}^{M}(2.5k)$, and $E\left[W_{T,2.5}^{M}\right]$ for $k = 1,2,\dots,\lfloor t/2.5 \rfloor$.

Taking $L = 30$, we get a grid of size 30 discretizing the set (0,30] into 30 equally spaced points from 1 to 30 for the preventive threshold M. Let M_i be the i-th value of M, which corresponds to the i-th value of the grid obtained previously, for $i = 1,2,\dots,40$.

For fixed $T = 2.5$ and for each fixed M_i, we obtain with MATLAB software 50000 simulation results for (R_1, I_1, W_d), where R_1 is the time to the first replacement (corrective or preventive), I_1 corresponds to the first maintenance action performed (corrective or preventive), and W_d denotes the downtime up to the first maintenance action. If the first maintenance action is a preventive maintenance, W_d equals 0 since the system is still working when the replacement is performed.

With these simulations and applying Monte Carlo method, we obtain $\tilde{P}_{R_{1,p}}^{M}(2.5k)$, $\tilde{P}_{R_{1,c}}^{M}(2.5k)$, $\tilde{P}_{R_1}^{M}(2.5k)$, and $\tilde{E}\left[W_{T,2.5}^{M}\right]$, which correspond to the estimations for $P_{R_{1,p}}^{M}(2.5k)$, $P_{R_{1,c}}^{M}(2.5k)$, $P_{R_1}^{M}(2.5k)$, and $E\left[W_{T,2.5}^{M}\right]$ for $k = 1,2,\dots,\lfloor t/2.5 \rfloor$ given by Equations (4), (5), (6), and (7), respectively.

For each M_i and $T = 2.5\, t.u.$, we estimate the expected cost in finite-time horizon at time t for $t \geq 2.5$ throughout the following recursive expression

$$\tilde{E}[C_{2.5}^{M}(t)] = \sum_{k=1}^{\lfloor t/2.5 \rfloor} \tilde{E}[C_{2.5}^{M}(t-2.5k)] \tilde{P}_{R_1}^{M}(2.5k) + \tilde{G}_{2.5}^{M}(t)$$

where

$$\tilde{G}_{2.5}^{M}(t) = \left(\lfloor t/2.5 \rfloor C_i(k-1) + C_d \tilde{E}\left[W_{\lfloor t/2.5 \rfloor}^{M}\right]\right) \left(1 - \sum_{k=1}^{\lfloor t/2.5 \rfloor} \tilde{P}_{R_1}^{M}(2.5k)\right) + \sum_{k=1}^{\lfloor t/2.5 \rfloor} \left(C_p + C_i(k-1)\right) \tilde{P}_{R_{1,p}}^{M}(2.5k) + \sum_{k=1}^{\lfloor t/2.5 \rfloor} \left(C_c + C_i(k-1)\right) \tilde{P}_{R_{1,c}}^{M}(2.5k) + \sum_{k=1}^{\lfloor t/2.5 \rfloor} C_d E\left[W_{2.5,k}^{M}\right] \tilde{P}_{R_{1,c}}^{M}(2.5k),$$

with initial conditions $\tilde{E}[C_{2.5}^{M}(0)] = 0$ and $\tilde{G}_{2.5}^{M}(0) = 0$.

Let $\tilde{E}^{*}\left[C_{2.5}^{M_i}(10)\right]$ be the expected cost rate in finite-time horizon at time $t_f = 10$ for $T = 2.5\, t.u.$ and variable M, that is $\tilde{E}^{*}\left[C_{2.5}^{M_i}(10)\right] = \tilde{E}\left[C_{2.5}^{M_i}(10)\right]/10$. *Figure 4* shows the value of $\tilde{E}^{*}[C_{2.5}^{M}(10)]$ for each value of M.

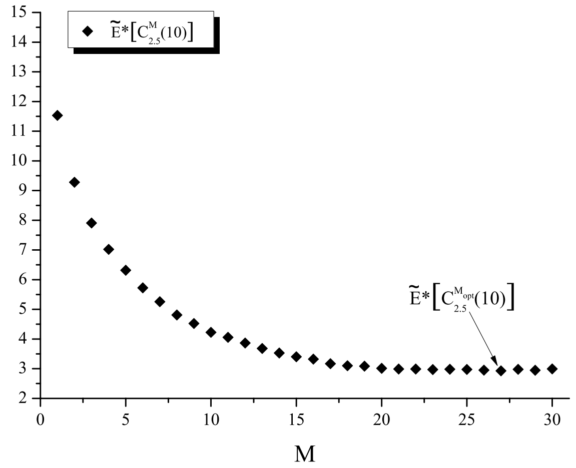

Figure 4. Expected cost rate in finite-time horizon at time $t_f = 10\, t.u.$ for different values of M

Based on *Figure 4*, the value of M which minimize the expected cost rate in finite-time horizon at time $t_f = 10\, t.u.$ is reached at $M_{opt} = 27$ with a expected cost rate of $2.9281\, m.u./t.u.$

Now, we focus on the influence of different model parameters on the solution. Firstly, a sensitivity analysis for the gamma process parameters are performed. Later, the parameters λ_1 and λ_2 are considered.

The values of the gamma process are modified under the following specifications:

$$\alpha_{(v_i\%)} = \alpha\left[1 + \frac{v_i}{100}\right], \quad \beta_{(v_j\%)} = \beta\left[1 + \frac{v_j}{100}\right], \qquad (10)$$

where v_i and v_j are the i-th and j-th positions of the vector $\boldsymbol{v} = (-10,-5,-1,0,1,5,10)$, respectively. Then, the value of the parameters α and β are simultaneous and independently modified for increasing and decreasing changes.

Let $\tilde{E}^{*}\left[C_{2.5,\alpha_{(v_i\%)},\beta_{(v_j\%)}}^{M_i}(10)\right]$ be the minimal value of the expected cost rate at time $t_f = 10$ for different values of M with fixed $T = 2.5\, t.u.$ obtained by varying simultaneously α and β according to the specifications given in (10). Then, a relative variation percentage $V_{\alpha_{(v_i\%)},\beta_{(v_j\%)}}(10)$ is defined as

$$\frac{\tilde{E}^*\left[C_{2.5}^{M_{opt}}(10)\right] - \tilde{E}^*\left[C^{M_i}_{2.5,\alpha_{(v_i\%)},\beta_{(v_j\%)}}(10)\right]}{\tilde{E}^*\left[C_{2.5}^{M_{opt}}(10)\right]},$$

where $\tilde{E}^*\left[C_{2.5}^{M_{opt}}(10)\right]$ represents the minimal expected cost rate calculated previously with initial parameters as $\tilde{E}^*\left[C_{2.5}^{M_{opt}}(10)\right] = 2.9281\ \mathrm{m.u./t.u.}$

For fixed i and j, $V_{\alpha_{(v_i\%)},\beta_{(v_j\%)}}(10)$ measures the relative difference between the current optimal cost and the optimal cost that has been calculated by using the modified parameter values. If this quantity is multiplied by 100, the result is expressed in percentage. Values closer to zero show a less influence on the solution.

Table 1 shows the relative variation percentages with a shaded gray scale. Each cell represents $V_{\alpha_{(v_i\%)},\beta_{(v_j\%)}}(10)$ multiplied by 100. Darker colors of cells denote a higher relative variation percentage.

Table 1. Relative variation percentages for the expected cost rate for the gamma process parameters

	$\beta_{(\ 10\%)}$	$\beta_{(\ 5\%)}$	$\beta_{(\ 1\%)}$	β	$\beta_{(1\%)}$	$\beta_{(5\%)}$	$\beta_{(10\%)}$
$\alpha_{(\ 10\%)}$	3.1850	1.4194	4.1122	5.8608	5.7754	8.0981	11.1164
$\alpha_{(\ 5\%)}$	6.3611	2.1973	1.5433	2.0027	3.1621	6.1781	8.8764
$\alpha_{(\ 1\%)}$	9.3426	4.3885	0.2975	0.2582	1.1294	3.7403	6.8389
α_1	10.1998	4.5869	1.3883	0.0000	0.3719	3.0351	6.7487
$\alpha_{(1\%)}$	10.9573	4.9571	1.4173	0.9173	0.1021	3.6911	6.1866
$\alpha_{(5\%)}$	14.1856	8.1353	4.0917	3.9234	2.5856	0.3791	4.6477
$\alpha_{(10\%)}$	17.3184	10.4040	8.1561	7.1620	5.4612	2.0764	1.2322

By modifying $\pm$ 1% around $\alpha = \beta = 0.1$, the relative variation percentages are small. The results also show that the relative variation percentages are lower in the diagonal of the table. That means, when the parameters α and β are modified in the same direction and magnitude.

Similarly, the values of the parameters λ_1 and λ_2 are modified according to the following specifications:

$$\lambda_{1(v_i\%)} = \lambda_1\left[1 + \frac{v_i}{100}\right],$$
$$\lambda_{2(v_j\%)} = \lambda_2\left[1 + \frac{v_j}{100}\right], \quad (11)$$

Let $\tilde{E}^*\left[C^{M_i}_{2.5,\lambda_{1(v_i\%)},\lambda_{2(v_j\%)}}(10)\right]$ be the minimal expected cost rate obtained by varying the parameters λ_1 and λ_2 simultaneously as in the scheme given in (11). Now, the relative variation percentage $V_{\lambda_{1(v_i\%)},\lambda_{2(v_j\%)}}(10)$ is given by

$$\frac{\tilde{E}^*\left[C_{2.5}^{M_{opt}}(10)\right] - \tilde{E}^*\left[C^{M_i}_{2.5,\lambda_{1(v_i\%)}\,\lambda_{2(v_j\%)}}(10)\right]}{\tilde{E}^*\left[C_{2.5}^{M_{opt}}(10)\right]}.$$

The relative variation percentages are presented in *Table 2.*

Table 2. Relative variation percentages for the expected cost rate for the parameters λ_1 and λ_2

	$\lambda_{2,(\ 10\%)}$	$\lambda_{2,(\ 5\%)}$	$\lambda_{2,(\ 1\%)}$	λ_2	$\lambda_{2,(1\%)}$	$\lambda_{2,(5\%)}$	$\lambda_{2,(10\%)}$
$\lambda_{1,(\ 10\%)}$	4.3506	3.9377	3.6078	3.0416	3.3862	2.6785	2.9063
$\lambda_{1,(\ 5\%)}$	3.4507	1.8743	1.5099	2.3274	2.3001	0.6380	0.4248
$\lambda_{1,(\ 1\%)}$	1.1246	1.2329	0.1356	0.1581	0.6079	0.6906	0.7640
λ_1	0.2193	1.4839	0.2172	0.0000	0.9918	1.4924	0.7831
$\lambda_{1,(1\%)}$	0.1004	0.1048	1.4719	0.2848	0.1216	1.3644	2.1382
$\lambda_{1,(5\%)}$	1.2025	2.9152	1.7250	2.5006	2.6273	2.1734	2.8247
$\lambda_{1,(10\%)}$	3.1973	3.6826	3.9350	5.0589	4.7949	4.5183	4.9165

Once again, by modifying $\pm$ 1% around $\lambda_1 = 0.01$ and $\lambda_2 = 0.1$, the relative variation percentages are small. Thus, the results show that the parameter λ_1 has greater effects on $V_{\lambda_{1(v_i\%)},\lambda_{2(v_j\%)}}(10)$ than the parameter λ_2.

5. Conclusions and further works

In this paper, a CBM strategy in finite-time horizon for a system is considered. This system is subject to a degradation process modelled under a gamma process and sudden shock processes following an NHPP. We consider that both causes of failure are dependent. This dependence is reflected in the fact that the system is more susceptible to external shocks when the deterioration level of the system reaches a certain threshold. In order to complete the study, a numerical example is proposed in order to illustrate the estimation of the expected cost rate in finite-time horizon by using a method which combines a recursive method and Monte Carlo simulation. In addition, the robustness of the solution is analyzed when we vary different parameters of the model.

A further possible extension of this work could be studying the expected cost rate in finite-time horizon varying the time between inspections instead of the preventive threshold. Other possible extension of this work is considering a system subject to multiple processes of deterioration completed with analyzing the associated standard deviation of the expected cost rate in finite-time horizon.

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