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A Markov model of a complex technical system operating in basic and emergency modes

Keywords

multi-state system, Markov process, failure/repair rate, component importance, transition intensity

Abstract

A complex technical system built of independent repairable components with constant failure and repair rates is examined. The system can operate in either basic or emergency mode, and its behavior is modeled by a three-state Markov process. It is demonstrated how to obtain closed formulas for the state probabilities of this process and the so-called importances of individual components to the inter-state transitions. Such an importance is defined as the probability that a component's failure/repair causes a transition between two given states of the modeling process. The obtained formulas allow to compute a number of reliability parameters characterizing the dynamics of the system's operation. The obtained results are illustrated by their application to an exemplary reliability block diagram that can be a model of a power supply network, a gas or oil pipeline system, etc.

1. Introduction

In this paper, a complex system built of independent repairable components with constant failure and repair rates is considered. The system can operate in either basic or emergency mode, and its functioning is modeled by a three-state Markov process on the state space {0,1,2}. The system is in state 1 if it operates in basic mode, in state 2 if it operates in emergency mode due to basic mode's failure, or in state 0 if both modes are failed. The paper's aim is to present analytical formulas for the system's key reliability parameters, derived by the author.
As an example, let us take a small power supply network whose reliability block diagram (RBD) is displayed in *Figure 1*. The boxes denoted $e_1,\ldots,e_8$ represent the network's components listed below.

e_1 – distribution company's network
e_2 – renewable source connected to e_5
e_3 – low voltage bus bar
e_4 – low voltage cut-off switch + low voltage cable line + low voltage cut-off switch
e_5 – load point (LP)
e_6 – transfer switch + low voltage cut-off switch
e_7 – low voltage cut-off switch + low voltage cable line + low voltage cut-off
e_8 – load point

Remark: + denotes the serial connection between elements of e_4, e_6, or e_7. The failure and repair rates of e_4, e_6, and e_7 can be found from (5) and (6), where S={0,1}, $0 <_d 1$.

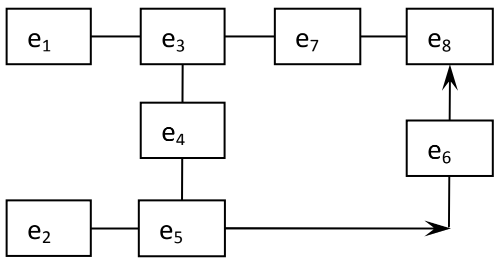

Figure 1. RBD of a small power supply network. Arrows show the end of emergency supply path to e_8

Let us shortly analyze the network's functioning with respect to the LP e_8 which operates in normal mode when all elements along the path (e_1, e_3, e_7, e_8) or (e_2, e_5, e_4, e_3, e_7, e_8) are operable. When the normal mode fails, which can be caused by e.g. a failure of e_3 or e_7, then e_8 is switched to emergency mode by the transfer switch in e_6, provided that all elements along the path (e_2, e_5, e_6, e_8) or (e_1, e_3, e_4, e_5, e_6, e_8) are operable. When the normal mode is restored then e_8 is switched back into normal mode. Obviously, it can happen that both the normal and emergency modes are failed, and then a power outage occurs at e_8. The functioning of e_8 can thus be modeled by a three-state stochastic process on the earlier defined state space.

We now give the outline of the paper. It is composed of four main sections numbered 2 through 5 and the 6-th concluding section. In section 2 the Markov property of the random process modeling the considered system's behavior is proved. The formulas for computing the system's key reliability parameters, i.e. the state probabilities and the inter-state transition intensities of the modeling process, are derived in sections 4 and 5 for which the theoretical background is presented in sections 2 and 3. The considerations of section 5 are based on the above given example, but can easily be extended to the general case. In section 6 it is shown how other parameters, characterizing the dynamics of switching between basic and emergency modes, can be obtained.

Readers interested in the reliability of power distribution systems (such a system serves as an example here) are referred to [2], while those needing an insight into the general theory of reliability – to [1] and [6].

2. General formulas for the transition intensities of a multistate system with two-state renewable components

Let us consider a multistate complex system composed of two-state components such that their failure-repair processes are two-state independent homogenous Markov chains. The system will be described by the following characteristics:

$\{e_i, 1 \le i \le n\}$ – set of the system's components

$J = \{1,\ldots,n\}$ – set of the components' indices

λ_i, μ_i – failure and repair rates of e_i

x_i – binary variable representing the state of e_i, i.e. $x_i=1/x_i=0$ if e_i is operable/failed

x – vector of the components' states, $x = [x_1,\ldots,x_n]$

$\{0,1\}^n$ – set of binary vectors of length n

$[x, 1_i]$, $[x, 0_i]$ – vector x whose i-th coordinate is set to 1 or 0

d(x, y) – number of coordinates in which vectors x and y differ (the Hamming distance)

$X_i(t)$ – state of e_i at time t (a random variable)

$p_i(t)$, $q_i(t)$ – state probabilities of $X_i(t)$, i.e. $p_i(t) = \Pr[X_i(t)=1]$, $q_i(t) = \Pr[X_i(t)=0]$

X(t) – vector of the components' states at time t, i.e. $X(t) = [X_1(t),\ldots,X_n(t)]$

$\Phi(x)$ – the system's structure function expressing the system's state in relation to the components' states

S – the discrete set of the system's states with the partial order transferred by Φ from the partial order in $\{0,1\}^n$, i.e.

$$(x, y \in \{0,1\}^n) \wedge (x < y) \Rightarrow \Phi(x) \le \Phi(y)$$

where < and ≤ denote the strong and weak precedence relations in $\{0,1\}^n$ and S. We adopt the usual partial order in $\{0,1\}^n$, i.e. x < y if d(x,y)>0 and $y_i - x_i = 1$ for each $x_i \ne y_i$.

$<_d$ – direct precedence relation in $\{0,1\}^n$ and S

Z(t) – the system's state at time t, i.e. $Z(t) = \Phi(X(t))$

Z – stochastic process $(Z_t, t\ge0)$, where $Z_t = Z(t)$

$\Lambda_{a\to b}(t)$ – intensity with which Z changes its state from a to b at time t (a transition intensity), defined as follows:

$$\Lambda_{a\to b}(t) = \\ = \lim_{\Delta t\to 0} \frac{1}{\Delta t}\Pr[Z(t+\Delta t) = b \mid Z(t) = a] \tag{1}$$

$\Pi_{a\to b}^{crit}(i)$ – set of binary vectors x such that $x_i=1$, $\Phi(x)=a$, and $\Phi(x, 0_i)=b$

$\Theta_{a\to b}^{crit}(i)$ – set of binary vectors x such that $x_i=0$, $\Phi(x)=a$, and $\Phi(x, 1_i)=b$

$I_{a\leftrightarrow b}(i)$ – importance of e_i to a transition between a and b, defined as follows:

$$I_{a\leftrightarrow b}(i) = \Pr\left[X \in \Pi_{a\to b}^{crit}(i) \mid X_i = 1\right] \\ = \Pr\left[X \in \Theta_{b\to a}^{crit}(i) \mid X_i = 0\right] \tag{2}$$

i.e. $I_{a\leftrightarrow b}(i)$ is the probability that the failure/repair of e_i causes a transition from a to b, given that e_i is operable/failed.

$\vee$ – the "Boolean" sum of real numbers from the [0, 1] interval, defined as $p_1 \vee p_2 = p_1 + (1 - p_1)p_2$

Remark 1: It was shown in [3] that

$$p_i(t) = \frac{\mu_i}{\lambda_i+\mu_i} + \frac{\lambda_i}{\lambda_i+\mu_i}\exp[-(\lambda_i+\mu_i)t] \tag{3}$$

$$q_i(t) = \frac{\lambda_i}{\lambda_i+\mu_i} - \frac{\lambda_i}{\lambda_i+\mu_i}\exp[-(\lambda_i+\mu_i)t] \tag{4}$$

Remark 2: If $x\in\Pi_{a\to b}^{crit}(i)$ then we say that x is critical to the transition from state a to b caused by e_i's failure. If, in turn, $x\in\Theta_{a\to b}^{crit}(i)$ then we say that x is critical to the transition from state a to b caused by e_i's renewal. These notions of criticality are generalizations of a path-vectors' or a cut-vectors' criticality for a two-state system (see [1]). Clearly, if $x \in \Pi_{a\to b}^{crit}(i)$, then $[x, 0_i] \in \Theta_{b\to a}^{crit}(i)$.

Remark 3: $I_{a\leftrightarrow b}(i)$ is a generalization of the Birnbaum importance for a two-state system. Various types of components' importances in multi-component systems are discussed in [5].

Now the main result of this section, which is a theorem giving the expressions for the transition intensities of the process Z, will be formulated.

Theorem 1. Let $a,b \in S$, $a \neq b$. Then $\Lambda_{a\to b} > 0$ only if $a <_d b$ or $a >_d b$, which means that direct transitions between a and b are only possible if one directly precedes the other. If $a <_d b$ then we have:

$$\Lambda_{a\to b}(t) = \frac{1}{\Pr[Z(t)=a]} \sum_{i\in I} \mu_i q_i(t) I_{a\leftrightarrow b}(i,t) \qquad (5)$$

$$\Lambda_{b\to a}(t) = \frac{1}{\Pr[Z(t)=b]} \sum_{i\in I} \lambda_i p_i(t) I_{a\leftrightarrow b}(i,t) \qquad (6)$$

Proof. The proof will appear in the extended version of this paper, being prepared for publication.

Remark: the formulas (5) and (6) are generalizations of the analogous ones which can be found in [3] and [4].

An important conclusion can be drawn from *Theorem 1*. If Φ is a structure function, and the partial order in S is transferred by Φ from $\{0,1\}^n$, then Z is a Markov process with the transition intensities given by (5) and (6). Indeed, it can be easily shown that Pr[Z(t)=a], Pr[Z(t)=b], and $I_{a\leftrightarrow b}(i)$, $i \in J$ are functions of $p_i(t)$, hence these intensities are functions of t, and do not depend on the history of Z before time t. Also, they converge to constant values as $t\to\infty$, because $p_i(t)$ converges to $\mu_i/(\lambda_i+\mu_i)$ as $t\to\infty$. Z is thus asymptotically homogenous.

3. A Markov model of a two-mode system

A key assumption about the considered system is that its components are independent, i.e. the time-to-failure (TTF) and time-to-repair (TTR) of any component do not depend on any other component's TTF or TTR. Clearly, this assumption may seem doubtful, because a component has to wait for repair if all maintenance teams are busy repairing other failed components, in which case the dependence of the component's TTR on the TTRs of other components is evident. However, if the components' failure rates are very small compared to their repair rates, i.e. $\lambda_i << \mu_i$, $i \in J$, which is often the case in practical situations, then the probability that a component fails when another component undergoes repair is close to zero. Moreover, if there are at least two maintenance teams, then repairs of two (or more) components can be performed simultaneously, if (notwithstanding the small failure rates) a component fails while another one is under repair. In consequence, the system's functioning can be approximately described by n independent two–state Markov chains, each being the failure-repair process of the respective component. Such approach directly leads to the construction of a Markov chain with 2^n states. Nevertheless, it occurs that in order to model the system's functioning as perceived by a user the number of states can be greatly reduced. Such a model will be constructed in this section. In the next section formulas for the transition intensities of the respective Markov chain will be derived.
To begin with, a detailed model of the considered system is presented in *Figure 2* in the form of inter-state transitions diagram. The meanings of individual states are given below the figure. This model takes into consideration each situation that can arise as a consequence of the fact that the system has two modes of operation.

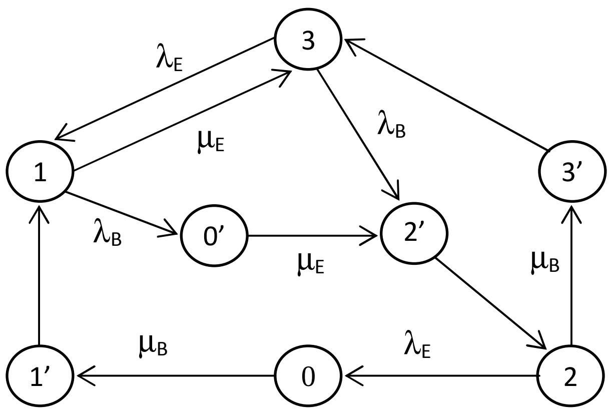

Figure 2. The detailed model of the system's operation

3 – both modes are operable, BM is active (EM has been switched to BM)
3' – both modes are operable, EM is active
1 – BM is active, EM is under repair
1' – BM is operable and inactive, EM is under repair
2 – BM is under repair, EM is active
2' – BM is under repair, EM is operable and inactive
0 – both modes are failed, BM is under repair, EM is awaiting repair
0' – both modes are failed, EM is under repair, BM is awaiting repair

Clearly, the stochastic process illustrated in *Figure 1* has the Markov property if the sojourn times in each state is exponentially distributed. However, *Theorem 1* cannot be applied here, because the process's state space is not an image of $\{0,1\}^n$ for any function defined on $\{0,1\}^n$. In particular, the transition from 1' to 1, 2' to 2, or 3' to 3 is not an effect of a component's state change, but that of switching between basic and emergency modes.

When there are two or more maintenance teams then repair of either mode can start immediately after its failure, and the states 0' and 0 can be merged into one state – 0. The resulting diagram is presented in *Figure 3*, and the remark regarding the application of *Theorem 1* still holds.

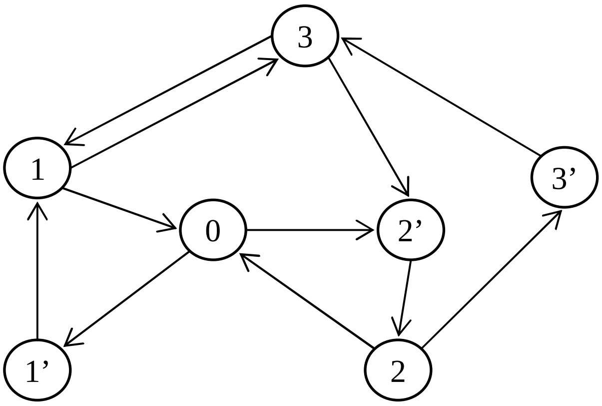

Figure 3. The model assuming no waiting times

Let us note that the transition chain 3→2'→2 is perceived by a system's user as the direct transition 3→2, i.e. from basic to emergency mode. Similarly, the transition chain 2→3'→3 is perceived as the direct transition 2→3, i.e. from emergency to basic mode. Further, the transition chains 0→1'→1 and 0→2'→2 are perceived as the direct transitions 0→1 and 0→2 respectively. In consequence, from a user's viewpoint, each of the states 1', 2', and 3' can be merged with the state 1, 2, or 3, respectively. The resulting diagram is shown in *Figure 4*.

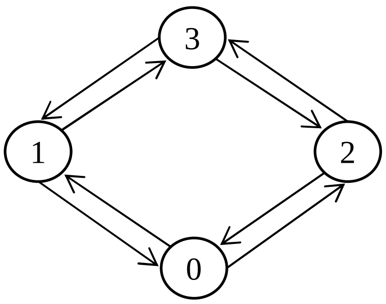

Figure4. The four-state model

As each transition involves a failure or repair of at least one component, the state space of the process illustrated in *Figure 4* is given by $S = \Phi(\{0,1\}^n) =$ $= \{0,1,2,3\}$, where Φ is the respective structure function. The partial order in S, transferred from that in $\{0,1\}^n$, is given by the following relations: $0 <_d 1$, $0 <_d 2$, $1 <_d 3$, $2 <_d 3$. We can thus find the inter-state transition intensities by applying *Theorem 1*.

Let us note that a user may not distinguish between the states 1 and 3, because in both cases the system operates in basic mode, and a transition between 1 and 3 does not cause a break in the system's operation, noticeable to a user. Thus our model can be further simplified, by merging the state 3 with 1, to the model presented in *Figure 5*.

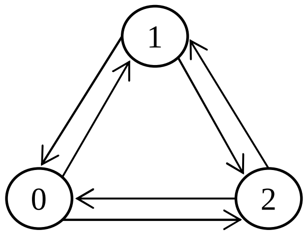

Figure 5. The three-state model

The state space of the process illustrated in *Figure 5* is given by $S = \Phi(\{0,1\}^n) = \{0,1,2\}$, where Φ is the respective structure function. The partial order in S, transferred from that in $\{0,1\}^n$, is given by the following relations: $0 <_d 1$, $0 <_d 2$, $2 <_d 1$. Thus, as in the previous case, we can apply *Theorem 1* in order to find the inter-state transition intensities, which will be done in the next section.

We conclude this chapter with a remark relevant to possible applications of the presented model. It can be assumed that the switching between basic and emergency modes is done instantly, i.e. times of transitions 1'→1, 2'→2, and 3'→3 are much shorter than the remaining transition times. Thus, in the three-state model, the transitions 1→2 and 2→1 are associated with short breaks in the system operation. In turn, the sojourn in state 0 is perceived as a long break, because it involves restoring and activating basic or emergency mode.

4. Formulas for the transition intensities of the three-state system

In this section, let Z be the three-state process illustrated in *Figure 5*. Let also $P_s(t) = \Pr[Z(t)=s]$, $s \in \{0,1,2\}$. We will now derive formulas for the transition intensities of Z. This task will be made easier by using the intensities of transitions of two-state processes obtained from Z by aggregating the states 0+2 or 1+2. For simpler notation, we will omit the variable t where no confusion arises. These intensities are defined as follows:

$$\Lambda_{1\to 0+2}(t) =$$

$$= \lim_{u \to 0} \frac{1}{u} \Pr[Z(t+u) \in \{0,2\} | Z(t) = 1] \quad (7)$$

$$\Lambda_{0+2\to 1}(t) =$$

$$= \lim_{u \to 0} \frac{1}{u} \Pr[Z(t+u) = 1 | Z(t) \in \{0,2\}] \quad (8)$$

$$\Lambda_{1+2\to 0}(t) =$$

$$= \lim_{u \to 0} \frac{1}{u} \Pr[Z(t+u) = 0 | Z(t) \in \{0,2\}] \quad (9)$$

$$\Lambda_{0\to 1+2}(t) =$$

$$= \lim_{u \to 0} \frac{1}{u} \Pr[Z(t+u) \in \{1,2\} | Z(t) = 0] \quad (10)$$

where + is the aggregation operator. The definition of a component's importance to an inter-state transition yields:

$$I_{1\leftrightarrow 0+2}(i) = P_1|_{p_i=1} - P_1|_{p_i=0} \quad (11)$$

$$I_{1+2\leftrightarrow 0}(i) = P_{1+2}|_{p_i=1} - P_{1+2}|_{p_i=0} \quad (12)$$

where $i \in J$, and $P_{1+2}(t) = \Pr[Z(t) \in \{1,2\}] = P_1(t)+P_2(t)$. From (5) and (6) we obtain:

$$\Lambda_{1\to 0+2} = \frac{1}{P_1} \sum_{i \in S} \lambda_i p_i I_{1\leftrightarrow 0+2}(i) \quad (13)$$

$$\Lambda_{0+2\to 1} = \frac{1}{1-P_1} \sum_{i \in S} \mu_i q_i I_{1\leftrightarrow 0+2}(i) \quad (14)$$

$$\Lambda_{1+2\to 0} = \frac{1}{P_{1+2}} \sum_{i \in S} \lambda_i p_i I_{1+2\leftrightarrow 0}(i) \quad (15)$$

$$\Lambda_{0\to 1+2} = \frac{1}{1-P_{1+2}} \sum_{i \in S} \mu_i q_i I_{1+2\leftrightarrow 0}(i) \quad (16)$$

The transition intensities of Z will be expressed using those defined by (13)-(16), and the importances $I_{1\to 2}(i)$, $i \in J$. However, the latter are not given by formulas as simple as (11)-(12). This is due to the fact that Z is not a two-state process. A method to compute $I_{1\to 2}(i)$, $i \in J$ will be presented in the next section.
As $2 <_d 1$, from (5) and (6) we get:

$$\boldsymbol{\Lambda_{2\to 1} = \frac{1}{P_2} \sum_{i \in I} \mu_i q_i I_{1\leftrightarrow 2}} \quad (17)$$

$$\Lambda_{1\to 2} = \frac{1}{P_1} \sum_{i \in I} \lambda_i p_i I_{1\leftrightarrow 2} \quad (18)$$

From (7), (8), and the law of total probability it follows that:

$$\Lambda_{1\to 0+2} = \Lambda_{1\to 0} + \Lambda_{1\to 2} \quad (19)$$

$$\Lambda_{0+2\to 1} = \frac{\Lambda_{0\to 1} P_0 + \Lambda_{2\to 1} P_2}{P_0 + P_2} \quad (20)$$

where $P_0(t) = \Pr[Z(t)=0] = 1 - P_1(t) - P_2(t)$. From (19) and (20) we have:

$$\Lambda_{1\to 0} = \Lambda_{1\to 0+2} - \Lambda_{1\to 2} \quad (21)$$

$$\Lambda_{0\to 1} = \frac{\Lambda_{0+2\to 1}(P_0 + P_2) - \Lambda_{2\to 1} P_2}{P_0} \quad (22)$$

It now remains to compute $\Lambda_{2\to 0}$ and $\Lambda_{0\to 2}$. By the same argument as just used we obtain:

$$\Lambda_{0\to 1+2} = \Lambda_{0\to 1} + \Lambda_{0\to 2} \quad (23)$$

$$\Lambda_{1+2\to 0} = \frac{\Lambda_{1\to 0} P_1 + \Lambda_{2\to 0} P_2}{P_1 + P_2} \quad (24)$$

The equalities (23) and (24) yield:

$$\Lambda_{0\to 2} = \Lambda_{0\to 1+2} - \Lambda_{0\to 1} \quad (25)$$

$$\Lambda_{2\to 0} = \frac{\Lambda_{1+2\to 0}(P_1 + P_2) - \Lambda_{1\to 0} P_1}{P_2} \quad (26)$$

where $\Lambda_{0\to 1}$ and $\Lambda_{1\to 0}$ are given by (22) and (21) respectively.
As can be seen, all transition intensities of Z are expressed by $P_1(t)$, $P_2(t)$, and $I_{1\leftrightarrow 0}(i,t)$, $i \in J$, which, in turn, are functions of $p_i(t)$, $i \in J$. Z is thus a Markov chain with time-dependent transition intensities. As follows from (3), each $p_i(t)$: $i \in J$ converges to $\mu_i/(\lambda_i+\mu_i)$ as $t \to \infty$, hence each $\Lambda_{a\to b}(t)$: $a,b \in \{0,1,2\}$ converges to a constant value. In consequence, Z is asymptotically homogenous.
A method to find $P_1(t)$, $P_2(t)$, and $I_{1\leftrightarrow 2}(i,t)$, $i \in J$, which have to be known in order to use the formulas (17)-(18), (21)-(22), and (25)-(26), is presented in the next section 5.

3. Computing P_1, P_2, and $I_{1\leftrightarrow 2}$ for the exemplary system

As can be expected, the method to compute the state probabilities P_1 and P_2, and the importances $I_{1\leftrightarrow 2}(i)$, $i \in J$ is based on analyzing the system's RBD. This method will be illustrated using the RBD of the exemplary system, shown in *Figure 1*.
We will first find the formulas for P_1 and P_2. Let us note that the events $\{x_6=0\}$, $\{x_6=1, x_5=0\}$, $\{x_6=1, x_5=1, x_4=0\}$, $\{x_6=1, x_5=1, x_4=1, x_3=0\}$,…, $\{x_6=1,\ldots, x_3=1, x_7=0\}$, and $\{x_6=1,\ldots, x_3=1, x_7=1\}$ are disjoint and exhaustive. From the law of total

probability and the rules for computing the reliabilities of series-parallel systems, we obtain:

$$\begin{aligned} P_{1+2} = [\, & q_7(p_1p_3p_4 \vee p_2)p_5p_6 \\ & + p_7q_3(p_2p_5p_6) \\ & + p_7p_3q_4(p_1 \vee p_2p_5p_6) \\ & + p_7p_3p_4q_5(p_1) \\ & + p_7p_3p_4p_5q_6(p_1 \vee p_2) \\ & + p_7p_3p_4p_5p_6(p_1 \vee p_2)\,] \cdot p_8 \end{aligned} \tag{27}$$

Reversing the order of components in the first "Boolean" sum, and transforming all the "Boolean" sums using the definition of the operator $\vee$, we obtain:

$$\begin{aligned} P_{1+2} = [\, & q_7(p_2 + q_2p_1p_3p_4)p_5p_6 \\ & + p_7q_3p_2p_5p_6 \\ & + p_7p_3q_4(p_1 + q_1p_2p_5p_6) \\ & + p_7p_3p_4q_5p_1 \\ & + p_7p_3p_4p_5(p_1 + q_1p_2)\,] \cdot p_8 \end{aligned} \tag{28}$$

From (28) and the RBD in *Figure 1* it follows that:

$$\begin{aligned} P_1 = [\, & p_7p_3q_4(p_1) \\ & + p_7p_3p_4q_5p_1 \\ & + p_7p_3p_4p_5(p_1 + q_1p_2)\,] \cdot p_8 \end{aligned} \tag{29}$$

$$\begin{aligned} P_2 = [\, & q_7(p_2 + q_2p_1p_3p_4)p_5p_6 \\ & + p_7q_3p_2p_5p_6 \\ & + p_7p_3q_4q_1p_2p_5p_6\,] \cdot p_8 \end{aligned} \tag{30}$$

We now pass to the computation of $I_{1\leftrightarrow 2}(i)$. As $2 <_d 1$, transitions from 2 to 1 are triggered by components' repairs. From the RBD in *Figure 1* it can be seen that (1,3,7,8) and (2,5,4,3,7,8) are the basic minimal path-sets, and (2,5,6,8) and (1,3,4,5,6,8) are the emergency minimal path-sets. Let us note that if $i \in \{5,6,8\}$ then e_i belongs to the both emergency path-sets, which means that $\Phi(x) \neq 2$ if $x_i=0$. In consequence $\Theta_{2\to 1}^{crit}(i) = \varnothing$, and

$$I_{1\leftrightarrow 2}(i) = 0;\ i = 5, 6, 8 \tag{31}$$

For $i \in \{1,2,3,4,7\}$ the formula for $I_{1\leftrightarrow 2}(i)$ is obtained by first selecting those components in the expression for P_2, which contain the variable q_i. Each vector x such that $x_i=0$ (e_i is failed) and $\Phi(x)=2$ (the system's state is 2) corresponds to one such component. Then for each selected component it is checked if the repair of e_i "opens" at least one basic path, all of which are "closed" before e_i's repair. If so, the component (after the removal of the variable q_i, and a possible further modification) is added to the expression for $I_{1\leftrightarrow 2}(i)$. Clearly, in order to obtain this expression, the variable q_i has to be deleted from each selected component of P_2, as $I_{1\leftrightarrow 2}(i)$ is a conditional probability provided that $x_i=0$.

For i=1 the selected component is $p_7p_3q_4q_1p_2p_5p_6p_8$. As it contains the variables p_3, p_7, and p_8, the repair of e_1 opens the basic path (1,3,7,8), hence

$$I_{1\leftrightarrow 2}(1) = p_7p_3q_4p_2p_5p_6p_8 \tag{32}$$

For i=2 the selected component is $q_7q_2p_1p_3p_4p_5p_6p_8$. However, the repair of e_2 does not open any basic path due to the presence of q_7 in the analyzed component, hence

$$I_{1\rightarrow 2}(2) = 0 \tag{33}$$

For i=3 the selected component is $p_7q_3p_2p_5p_6p_8$. The repair of e_3 opens (1,3,7,8) provided that e_1 is operable, or (2,5,4,3,7,8) provided that e_4 is operable. We thus have

$$\begin{aligned} I_{1\leftrightarrow 2}(3) &= p_7p_2p_5p_6p_8(p_1 \vee p_4) \\ &= p_7p_2p_5p_6p_8(p_1 + q_1p_4) \end{aligned} \tag{34}$$

For i=4 the selected component is $p_7p_3q_4q_1p_2p_5p_6p_8$. The repair of e_4 opens (2,5,4,3,7,8) (note that it cannot open (1,3,7,8) due to the presence of q_1), hence

$$I_{1\leftrightarrow 2}(4) = p_7p_3q_1p_2p_5p_6p_8 \tag{35}$$

For i=7 the selected components are is $q_7p_2p_5p_6p_8$ and $q_7q_2p_1p_3p_4p_5p_6p_8$. In case of the first component, the repair of e_7 opens (1,3,7,8) if e_1 and e_3 are operable, or (2,5,4,3,7,8) if e_3 and e_4 are operable. In case of the second component the repair of e_7 opens (1,3,7,8) only (due to the presence of q_2). In consequence

$$\begin{aligned} I_{1\leftrightarrow 2}(7) = {} & p_2p_5p_6p_8(p_1 + q_1p_4)p_3 \\ & + q_2p_1p_3p_4p_5p_6p_8 \end{aligned} \tag{36}$$

6. Conclusion

A method to compute the state probabilities and inter-state transition intensities for a complex system operating according to a three-state Markov model has been presented. This method, appropriately modified, can be applied to a broad spectrum of Markov-modeled multi-state systems. In particular, the systems that fulfill the assumptions of *Theorem 1* are eligible. The computation of state probabilities and transition importances was presented in section 5 for the exemplary system, but it can easily be generalized to any system whose each path-set (obtained from its RBD) corresponds to one of the system's modes of operations (or states).

The key reliability parameters, i.e. the inter-state transition intensities can be used to obtain other characteristics of the system's behavior. Let us adopt the following definitions:

L_j – the mean sojourn time in the state j, $j \in \{0,1,2\}$

$N_{j\to k}(u)$ – the mean number of times the system changes its state from j to k in a time interval of length u.

$N^{long}(u)$ – the average number of long brakes in the system's operation, resulting from failures of the both modes

$N^{short}(u)$ – the average number of short breaks in the system's operation, caused by switching between the both modes

It can be simply shown that

$$L_j = \left(\sum_{k\in\{0,1,2\},k\neq j} \Lambda_{j\to k}\right)^{-1} \quad (37)$$

$$N_{j\to k}(u) = uA_j\Lambda_{j\to k} \quad (38)$$

$$N^{long}(\mathrm{u}) = N_{1\to 0}(u) + N_{2\to 0}(u) \quad (39)$$

$$N^{short}(\mathrm{u}) = N_{1\to 2}(u) + N_{2\to 1}(u) \quad (40)$$

The above defined characteristics are particularly important for the reliability analysis of power distribution networks.

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