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Reliability analysis of multistate series systems with dependent components

Keywords

reliability, dependent components, local load sharing, ageing components, multistate series system

Abstract

In the paper an approach to the reliability analysis of multistate series systems assuming their components dependency according to the local load sharing rule is presented. There is introduced the local load sharing model of components dependency (of departures from the reliability states subsets) and the reliability function of a multistate series system with dependent components is determined. These results are also presented in case system components have exponential reliability functions and in this case basic reliability characteristics are given.

1. Introduction

In reliability analysis the independence of system components is often assumed. This assumption means that failure of a component has no influence on the remaining surviving components. However, in many real technical systems after failure of any system components its load is transmitted to the remaining surviving components. Then, dependencies among system's components have significant influence on the reliability of a system. Thus, modeling systems with interdependent components is an important issue.

As emphasized by the authors of [16], a key factor of load sharing models is the rule that indicates how the failure rates of surviving components change after some components fail. Commonly used local load-sharing rules are equal, local and monotone load-sharing. The classical model of equal load sharing has been studied early by Daniels [6] and Smith [17], [18], and later by Pradhan et al. [14]. Kvam and Pena [12] have analyzed equal load-sharing model, in which the failure rates of the working components change uniformly after each failure within the system, but the magnitude of the change is unknown. Local load sharing rule, more realistic for some systems, in which the load on the failed component is transferred to its adjacent components and the load on the surviving components is directly proportional to their distance from the failed component was introduced by Harlow and Phoenix [7], [8] and further analyzed by Phoenix and Smith [13]. The monotone load-sharing rule dictates that the load on any individual component is non-decreasing as other components within the system fail. In [16] there is considered parallel system, in which some of components follow a constant failure rate and the remaining follow a linearly increasing failure rates. There can be found different models of failure dependency of system components as well as different approaches to this problem, both analytical and simulation [9], [15].

Authors in [22] develop a model for a load sharing system where an operator dispatches work load to components in a manner that manages their degradation. With this approach to the analysis of systems with dependent components taking into account the time to degradation failure of a system, and by estimating the probability of system's failure, it is possible to obtain optimal designs to minimize the long run average cost of a system.

This issue applies to both two-state systems, where dependencies of component failures are observed, and multistate systems, where dependencies of departures from the reliability state subsets are considered [1], [2]. This paper focuses on second of these cases. In reliability analysis of multistate systems with dependent components, after decreasing the reliability state by one of components, inside interactions among remaining components may cause further components reliability states decrease. A multistate approach [10], [19]-[21] to the reliability analysis of systems with dependent components assuming equal load sharing model has been presented in [3], [4]. Here, we consider local load sharing model of dependency for systems composed of degrading components, which allows to distinguish system's inside reliability states interactions.

2. Local load sharing model of components dependency

We suppose as in [10] and [11] that all components and a system under consideration have the reliability state set $\{0,1,...,z\}$, $z \geq 1$, where the state 0 is the worst and the state z is the best. The state of a system and components degrades with time without repair. Further, we consider a multistate series system, defined in [10] and [11], composed of n ageing and independent components with the reliability functions of its components

$$R_i(t,\cdot) = [R_i(t,0), R_i(t,1),...,R_i(t,z)],\ t \geq 0,\ i = 1,2,...,n, \tag{1}$$

where

$$R_i(t,u) = P(T_i(u) > t),\ i = 1,\dots,n, \tag{2}$$

and $T_i(u)$, $i = 1,2,...,n$, are independent random variables representing lifetimes of components E_i in the reliability state subset $\{u,u+1,...,z\}$, $u = 0,1,...,z$. Similarly, as in [10] and [11], we define the multistate reliability function of a system as a vector

$$\boldsymbol{R}(t,\cdot) = [\boldsymbol{R}(t,0), \boldsymbol{R}(t,1),\dots,\boldsymbol{R}(t,z)],\ t \geq 0, \tag{3}$$

where its coordinates

$$\boldsymbol{R}(t,u) = P(T(u) > t),\ u = 0,1,...,z,$$

and $T(u)$ is a random variable representing the lifetime of a system in the reliability state subset $\{u,u+1,...,z\}$, $u = 0,1,...,z$. Under this definition $\boldsymbol{R}(t,0) = 1$ for $t \geq 0$ and further we will use 1 instead of $\boldsymbol{R}(t,0)$.

Taking into account the dependence of components, we assume that after changing the reliability state subset by one of system components to the worse reliability state subset, lifetimes of remaining system components in the reliability state subsets decrease mostly for neighbour components in first line, then less for neighbour components in second line and so on. Further, we call this rule of components dependency a local load sharing (LLS) rule. More exactly, in this rule if the system component E_j, $j = 1,\dots,n$, gets out of the reliability state subset $\{u,u+1,\dots,z\}$, $u = 1,2,\dots,z$, the reliability parameters of remaining system components are changing dependently of the distance from the component that has got out of the reliability state subset $\{u,u+1,\dots,z\}$, $u = 1,2,\dots,z$, expressed by index d. The meaning of d is illustrated in *Figure 1*.

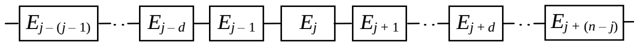

Figure 1. The meaning of the distance d

We denote by $E[T_i(u)]$ and $E[T_{i/j}(u)]$, $i = 1,2,...,n$, $j = 1,2,...,n$, $u = 1,2,\dots,z$, the mean values of system components lifetimes $T_i(u)$ and $T_{i/j}(u)$, respectively, before and after departure of one fixed component E_j, $j = 1,\dots,n$, from the reliability state subset $\{u,u+1,\dots,z\}$, $u = 1,2,\dots,z$. With this notation, in considered local load sharing rule, the components lifetimes and the mean values of components lifetimes in the reliability state subset $\{\upsilon,\upsilon+1,\dots,z\}$, $\upsilon = u,u-1,\dots,1$, $u = 1,2,\dots,z$, are decreasing according to the following formula:

$$E[T_{i/j}(\upsilon)] = q(\upsilon, d_{ij}) \cdot E[T_i(\upsilon)],\quad i = 1,\dots,n,\ j = 1,\dots,n,\ \upsilon = u, u-1,\dots,1, \tag{4}$$

where $q(\upsilon, d_{ij})$, $0 < q(\upsilon, d_{ij}) \leq 1$, $i = 1,\dots,n$, $j = 1,\dots,n$, and $q(\upsilon,0) = 1$, for $\upsilon = u,u-1,\dots,1$, $u = 1,2,\dots,z-1$, are non-increasing coefficients of components' distance $d_{ij} = |i-j|$ from the component that has got out of the reliability state subset $\{u,u+1,\dots,z\}$, $u = 1,2,\dots,z$.

We denote by

$$R_{i/j}(t,\upsilon) = P(T_{i/j}(\upsilon) > t),\ i = 1,\dots,n,\ j = 1,\dots,n, \tag{5}$$

for $\upsilon = u,u-1,\dots,1$, $u = 1,2,\dots,z-1$, the coordinates of the reliability function

$$R_{i/j}(t,\cdot) = [1, R_{i/j}(t,1),\dots, R_{i/j}(t,u)], \tag{6}$$

of a system component E_i, $i = 1,\dots,n$, after departure of the jth component E_j, $j = 1,\dots,n$, from the reliability state subset $\{u,u+1,\dots,z\}$, $u = 1,2,\dots,z$.

The system reliability function after its departure from the reliability state subset $\{u+1,u+2,\dots,z\}$, $u = 1,2,\dots,z$, we denote by

$$\breve{\boldsymbol{R}}(t,\cdot) = [1, \breve{\boldsymbol{R}}(t,1), \ldots, \breve{\boldsymbol{R}}(t,z)],\ t \geq 0, \tag{7}$$

where $\breve{\boldsymbol{R}}(t,u),\ u = 1,2,\ldots,z-1,$ are the coordinate reliability functions of the system with components having changed their reliability functions according to the local sharing rule expressed in (5)-(6) and $\breve{\boldsymbol{R}}(t,z) = \boldsymbol{R}(t,z),$ where $\boldsymbol{R}(t,z)$ is the zth coordinate reliability function of the system before the component departure defined by (3).

3. Reliability of multistate series system with dependent components

In [5] the below formulated theorem, concerned with reliability of a series system with dependent components, is proofed.

Proposition 1. If in a multistate series system components are dependent according to the local load sharing rule and have reliability functions given by (1)-(2), then its reliability function is given by the vector

$$\boldsymbol{R}_{LLS}(t,\cdot) = [1, \boldsymbol{R}_{LLS}(t,1), \ldots, \boldsymbol{R}_{LLS}(t,z)],\ t \geq 0, \tag{8}$$

with the coordinates

$$\boldsymbol{R}_{LLS}(t,u) = \prod_{i=1}^{n} R_i(t,u+1) \cdot \prod_{i=1}^{n} R_i(t,u)$$

$$+ \int_0^t \sum_{j=1}^{n} [f_j(a,u+1) \cdot \prod_{\substack{i=1 \\ i\neq j}}^{n} R_i(a,u+1) \cdot \prod_{i=1}^{n} R_{i/j}(t-a,u)$$

$$\cdot \prod_{i=1}^{n} R_i(a,u)]da,\ u = 1,2,\ldots,z-1, \tag{9}$$

$$\boldsymbol{R}_{LLS}(t,z) = \prod_{i=1}^{n} R_i(t,z), \tag{10}$$

where:

$R_i(t,u+1) -$ the reliability function coordinate of a component E_i, $i = 1,\ldots,n$, i.e. the probability that its lifetime in the reliability state subset $\{u+1,\ldots,z\}$, $u = 1,2,\ldots,z-1$, is greater than t,

$R_i(t,u) -$ the reliability function coordinate of a component E_i, $i = 1,\ldots,n$, i.e. the probability that its lifetime in the reliability state subset $\{u,u+1,\ldots,z\}$, $u = 1,2,\ldots,z$, is greater than t,

$f_j(t,u+1) -$ the density function coordinate of a component E_j, $j = 1,\ldots,n$, lifetime in the reliability state subset $\{u+1,\ldots,z\}$, $u = 1,2,\ldots,z-1$,

$R_{i/j}(t,u) -$ the reliability function coordinate of a component E_i, $i = 1,\ldots,n$, i.e. the probability that its lifetime in the reliability state subset $\{u,u+1,\ldots,z\}$, $u = 1,2,\ldots,z-1$, after departure from the reliability state subset $\{u+1,\ldots,z\}$, $u = 1,2,\ldots,z-1$, by the component E_j, $j = 1,\ldots,n$, is greater than t.

We can notice that, if the non-increasing coefficients $q(u,d_{ij})$, $i = 1,\ldots,n$, $j = 1,\ldots,n$, $u = 1,2,\ldots,z-1$, of distance $d_{ij} = |i-j|$ between component E_i, $i = 1,\ldots,n$, and component E_j, $j = 1,\ldots,n$, that has got out of the reliability state subset $\{u+1,\ldots,z\}$, $u = 1,2,\ldots,z-1$, is equal to 1, then the considered system becomes a series system with independent components. We can notice that if this condition holds, then $R_{i/j}(t,u) = R_i(t,u)$, $i = 1,\ldots,n$, $j = 1,\ldots,n$, i.e. $\breve{\boldsymbol{R}}(t,u) = \boldsymbol{R}(t,u)$, $u = 1,2,\ldots,z-1$, and the formula (9) takes form of the reliability function coordinate of the multistate series system with independent components for $u = 1,2,\ldots,z-1$.

More exactly, we obtain

$$\boldsymbol{R}_{LLS}(t,u) = \prod_{i=1}^{n} R_i(t,u+1) \cdot \prod_{i=1}^{n} R_i(t,u)$$

$$+ \int_0^t \sum_{j=1}^{n} [f_j(a,u+1) \cdot \prod_{\substack{i=1 \\ i\neq j}}^{n} R_i(a,u+1) \cdot \prod_{i=1}^{n} R_i(t-a,u)$$

$$\cdot \prod_{i=1}^{n} R_i(a,u)]da = \prod_{i=1}^{n} R_i(t,u+1) \cdot \prod_{i=1}^{n} R_i(t,u)$$

$$+ \int_0^t \sum_{j=1}^{n} [f_j(a,u+1) \cdot \prod_{\substack{i=1 \\ i\neq j}}^{n} R_i(a,u+1) \cdot \prod_{i=1}^{n} R_i(t,u)]da$$

$$= [\prod_{i=1}^{n} R_i(t,u+1) + \int_0^t \sum_{j=1}^{n} [f_j(a,u+1) \prod_{\substack{i=1 \\ i\neq j}}^{n} R_i(a,u+1)]da]$$

$$\cdot \prod_{i=1}^{n} R_i(t,u),\ u = 1,2,\ldots,z-1. \tag{11}$$

If one of n ageing system components E_j, $j = 1,\ldots,n$, gets out of the reliability state subset $\{u+1,\ldots,z\}$, $u = 1,2,\ldots,z-1$, in a time interval $<a, a+da)$ with the probability $f_j(a,u+1)da$, assuming that remaining system components E_i, $i = 1,\ldots,n$, $i \neq j$, have not left the reliability state subset $\{u+1,\ldots,z\}$, $u = 1,2,\ldots,z-1$, during time a with the probability $R_i(a,u+1)$, from the definition of a multistate series system given in [10] and [11], the whole system gets out of this reliability state subset. Then, from following formula

$$\sum_{j=1}^{n}[f_j(a,u+1)\cdot\prod_{\substack{i=1\\i\neq j}}^{n}R_i(a,u+1)] = \boldsymbol{f}(a,u+1), \quad (12)$$

for $0 < a < t,\ u = 1,2,\dots,z-1,$ where $\boldsymbol{f}(a,u+1)$ is the density function of a system lifetime in the reliability state subset $\{u+1,\dots,z\}$, $u = 1,2,\dots,z-1$ and considering formula for the reliability function coordinate $\boldsymbol{R}(t,u+1)$ of a multistate series system given by

$$\prod_{i=1}^{n}R_i(t,u+1) = \boldsymbol{R}(t,u+1),\ u = 1,2,\dots,z-1, \quad (13)$$

we get

$$\boldsymbol{R}_{LLS}(t,u) = [\boldsymbol{R}(t,u+1) + \int_0^t \boldsymbol{f}(a,u+1)da]\cdot\prod_{i=1}^{n}R_i(t,u)$$

$$= [\boldsymbol{R}(t,u+1) + \boldsymbol{F}(t,u+1)]\cdot\prod_{i=1}^{n}R_i(t,u)$$

$$= \prod_{i=1}^{n}R_i(t,u),\ u = 1,2,\dots,z-1. \quad (14)$$

Taking into account the result (14) and the formula (10), in case $q(u,d_{ij}) = 1$, $i = 1,\dots,n$, $j = 1,\dots,n$, $u = 1,2,\dots,z-1$, we get $\boldsymbol{R}_{LLS}(t,u) = \boldsymbol{R}(t,u)$, $u = 1,2,\dots,z$ and we conclude that formulae (8)-(10) present the reliability function of a multistate series system with independent components.
Further, we assume that components E_i, $i = 1,\dots,n$, have exponential reliability functions

$$R_i(t,\cdot) = [1, R_i(t,1),\dots,R_i(t,z)],\ t \geq 0, \quad (15)$$

with the coordinates

$$R_i(t,u) = \exp[-\lambda_i(u)t],\ u = 1,2,\dots,z, \quad (16)$$

where $\lambda_i(u)$, $\lambda_i(u) \geq 0$, $i = 1,\dots,n$, are components intensities of departure from the reliability state subset $\{u,u+1,\dots,z\}$, $u = 1,2,\dots,z$. Then according to the well known relationship between the lifetime mean value in this reliability state subset and the intensity of departure from this reliability state subset of the form

$$E[T_i(u)] = \frac{1}{\lambda_i(u)},\ u = 1,2,\dots,z, \quad (17)$$

we get the formula for the intensities $\lambda_{i/j}(\upsilon)$, $i = 1,\dots,n$, $j = 1,\dots,n$, $\upsilon = u,u-1,\dots,1$, of components departure from this reliability state subset after departure of the jth component E_j, $j = 1,\dots,n$. Namely, from formula (4), we obtain

$$\lambda_{i/j}(\upsilon) = \frac{\lambda_i(\upsilon)}{q(\upsilon,d_{ij})},\ \upsilon = u,u-1,\dots,1, \quad (18)$$

where $0 < q(\upsilon,d_{ij}) \leq 1$, $i = 1,\dots,n$, $j = 1,\dots,n$, and $q(\upsilon,0) = 1$, for $\upsilon = u,u-1,\dots,1$, $u = 1,2,\dots,z-1$, are non-increasing coefficients of components' distance $d_{ij} = |i-j|$ from the component that has got out of the reliability state subset $\{u,u+1,\dots,z\}$, $u = 1,2,\dots,z$.
In case system components have exponential reliability functions from *Proposition 1* we can obtain the following conclusion in a form of *Proposition 2*.
Proposition 2. If in a multistate series system components are dependent according to the local load sharing rule and have exponential reliability functions given by (15)-(16), then its reliability function is given by the vector

$$\boldsymbol{R}_{LLS}(t,\cdot) = [1, \boldsymbol{R}_{LLS}(t,1),\dots,\boldsymbol{R}_{LLS}(t,z)],\ t \geq 0, \quad (19)$$

with the coordinates

$$\boldsymbol{R}_{LLS}(t,u) = \exp[-[\sum_{i=1}^{n}\lambda_i(u+1) + \sum_{i=1}^{n}\lambda_i(u)]t]$$
$$+\sum_{j=1}^{n}\frac{\lambda_j(u+1)}{\sum_{i=1}^{n}\frac{\lambda_i(u)}{q(u,d_{ij})} - \sum_{i=1}^{n}\lambda_i(u) - \sum_{i=1}^{n}\lambda_i(u+1)}$$

$$\cdot[\exp[-[\sum_{i=1}^{n}\lambda_i(u+1) + \sum_{i=1}^{n}\lambda_i(u)]t]$$

$$-\exp[-\sum_{i=1}^{n}\frac{\lambda_i(u)}{q(u,d_{ij})}t]],\ u = 1,2,\dots,z-1, \quad (20)$$

$$\boldsymbol{R}_{LLS}(t,z) = \exp[-\sum_{i=1}^{n}\lambda_i(z)t]. \quad (21)$$

Corollary 1. If in a multistate series system components are dependent according to the local load sharing rule and have exponential reliability functions given by (15)-(16), then its mean lifetime in the reliability state subset $\{u,u+1,\dots,z\}$, $u = 1,2,\dots,z$, is given by

$$\mu_{LLS}(u) = \frac{1}{\sum_{i=1}^{n}\lambda_i(u+1) + \sum_{i=1}^{n}\lambda_i(u)}$$

$$+\sum_{j=1}^{n}\frac{\lambda_j(u+1)}{\sum_{i=1}^{n}\frac{\lambda_i(u)}{q(u,d_{ij})}-\sum_{i=1}^{n}\lambda_i(u+1)-\sum_{i=1}^{n}\lambda_i(u)}$$

$$\cdot[\frac{1}{\sum_{i=1}^{n}\lambda_i(u+1)+\sum_{i=1}^{n}\lambda_i(u)}-\frac{1}{\sum_{i=1}^{n}\frac{\lambda_i(u)}{q(u,d_{ij})}}] \qquad (22)$$

for $u=1,2,\ldots,z-1,$

$$\mu_{LLS}(z)=\frac{1}{\sum_{i=1}^{n}\lambda_i(z)}, \qquad (23)$$

and the standard deviation of the system sojourn time in the reliability state subset $\{u,u+1,\ldots,z\}$, $u = 1,2,\ldots,z$, is given by

$$\sigma_{LLS}(u)=\sqrt{n_{LLS}(u)-[\mu_{LLS}(u)]^2}, \qquad (24)$$

for $u=1,2,\ldots,z-1,$ where

$$n_{LLS}(u)=\frac{2}{[\sum_{i=1}^{n}\lambda_i(u+1)+\sum_{i=1}^{n}\lambda_i(u)]^2}$$

$$+2\sum_{j=1}^{n}\frac{\lambda_j(u+1)}{\sum_{i=1}^{n}\frac{\lambda_i(u)}{q(u,d_{ij})}-\sum_{i=1}^{n}\lambda_i(u+1)-\sum_{i=1}^{n}\lambda_i(u)}$$

$$\cdot[\frac{1}{[\sum_{i=1}^{n}\lambda_i(u+1)+\sum_{i=1}^{n}\lambda_i(u)]^2}-\frac{1}{[\sum_{i=1}^{n}\frac{\lambda_i(u)}{q(u,d_{ij})}]^2}] \qquad (25)$$

for $u=1,2,\ldots,z-1$ and

$$\sigma_{LLS}(z)=\frac{1}{\sum_{i=1}^{n}\lambda_i(z)}. \qquad (26)$$

4. Conclusion

In this paper, a model considering the influence of systems' inside-dependences on their reliability is presented. In reliability analysis of multistate ageing systems, it seems natural to assume that changes of reliability states of system components may influence remaining system components and cause their reliability characteristics worsening. In such systems the increased load caused by system component's failure i.e. changing the reliability state subset by component to the worse reliability state subset, may cause the decrease of lifetimes in this reliability state subset of remaining system components.

We have introduced a model of load sharing for multistate series systems with dependent components. The reliability function of a multistate series system under assumption that its components are dependent according to the local load sharing rule is determined and basic reliability characteristics are given in case system components have exponential reliability functions. The extension of these results to other complex systems is in preparation.

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References

[1] Blokus-Roszkowska, A. (2007). On component failures' dependency influence on system's lifetime. *Int J of Reliab, Quality and Safety Eng. Special Issue: System Reliability and Safety* 14(6), 1-19.

[2] Blokus-Roszkowska, A. (2007). *Reliability analysis of homogenous large systems with component dependent failures*. PhD Thesis, Gdynia Maritime University – System Research Institute Warsaw, (*in Polish*).

[3] Blokus-Roszkowska, A. & Kołowrocki, K. (2014). Reliability analysis of complex shipyard transportation system with dependent components. *Journal of Polish Safety and Reliability Association, Summer Safety and Reliability Seminars* 5(1), 21-31.

[4] Blokus-Roszkowska, A. & Kołowrocki, K. (2014). Reliability analysis of ship-rope transporter with dependent components. *Proc. European Safety and Reliability Conference – ESREL 2014*, Wrocław, 255-263.

[5] Blokus-Roszkowska, A. & Kołowrocki, K. (2015). Reliability analysis of multistate series systems with dependent components. *Applied Mathematics and Computation*, submitted.

[6] Daniels, H. E. (1945). The statistical theory of the strength of bundles of threads. I. *Proc. Roy. Soc. Ser. A.* 183, 404-435.

[7] Harlow, D. G. & Phoenix, S. L. (1978). The chain-of-bundles probability model for the strength of fibrous materials. *Journal of Composite Materials* 12, 195-214.
[8] Harlow, D .G. & Phoenix, S. L. (1982). Probability distribution for the strength of fibrous materials under local load sharing. *Adv. Appl. Prob.* 14, 68-94.
[9] Jain, M. & Gupta, R. (2012). Load sharing M-out of-N: G system with non-identical components subject to common cause failure. *International Journal of Operations Research* 4(5), 586–605.
[10] Kołowrocki, K. (2014). *Reliability of Large and Complex Systems.* Elsevier.
[11] Kołowrocki, K. & Soszyńska-Budny, J. (2011). *Reliability and Safety of Complex Technical Systems and Processes: Modeling – Identification – Prediction – Optimization*. Springer.
[12] Kvam, P.H. & Pena, E.A. (2005). Estimating load-sharing properties in a dynamic reliability system. *J Am Stat Assoc.* 1(100), 262-272.
[13] Phoenix, S.L. & Smith, R.L. (1983). A comparison of probabilistic techniques for the strength of fibrous materials under local load sharing among fibres. *Int. J. Solids Struct.* 19(6), 479-496.
[14] Pradhan, S., Hansen, A. & Chakrabarti, B.K. (2010). Failure processes in elastic fiber bundles. *Rev. Mod. Phys.* 82, 499-555.
[15] Singh, B. & Gupta, P.K. (2012). Load-sharing system model and its application to the real data set. *Mathematics and Computers in Simulation* 82(9), 1615–1629.
[16] Singh, B., Sharma, K.K. & Kumar, A. (2008). A classical and Bayesian estimation of a *k* components load sharing parallel system. *Computational Statistics and Data Analysis* 52, 5175–5185.
[17] Smith, R.L. (1983). Limit theorems and approximations for the reliability of load sharing systems. *Adv. Appl. Prob.* 15, 304-330.
[18] Smith, R.L. (1982). The asymptotic distribution of the strength of a series-parallel system with equal load sharing. *Ann. of Prob.* 10, 137-171.
[19] Xue, J. (1985). On multi-state system analysis. *IEEE Trans on Reliab.* 34, 329-337.
[20] Xue, J. & Yang, K. (1995). Dynamic reliability analysis of coherent multi-state systems. *IEEE Trans on Reliab.* 4(44), 683-688.
[21] Xue, J. & Yang, K. (1995). Symmetric relations in multi-state systems. *IEEE Trans on Reliab* 4(44), 689-693.
[22] Zhisheng, Y., Revie, M. & Walls, L. (2014). A Load Sharing System Reliability Model With Managed Component Degradation. *IEEE Trans. on Reliab.* 63(3), 721–730.