Operation and reliability optimization of complex technical systems

Keywords
reliability, multistate system, ageing, operation process, optimization, exemplary application

Abstract
Modelling and prediction of the operation and reliability of technical multistate ageing systems related to their operation processes called complex systems are briefly presented and applied to prediction of the operation processes and reliability characteristics of an exemplary complex non-homogeneous system composed of a series-parallel and a series-“m out of l” subsystems linked in series, changing its reliability structure and its components reliability parameters at variable operation conditions. Further, the linear programming is proposed to the operation and reliability optimization of complex technical systems operating at variable operation conditions. The method consists in determining the optimal values of limit transient probabilities at the system operation states that maximize the system lifetimes in the reliability state subsets. The proposed method is practically applied to the operation and reliability optimization of the considered exemplary complex system.

1. Introduction
The complexity of the systems’ operation processes and their influence on changing in time the systems’ structures and their components’ reliability parameters are very often met in real practice. Thus, the practical importance of an approach linking the system reliability models and the system operation processes models into an integrated general model in reliability assessment of real technical systems is evident. The convenient tools for analyzing these problems are semi-Markov modelling the systems’ operation processes [1]-[4], [12], [16]-[17], [20], [26] and multistate approach to the systems’ reliability evaluation [6], [10], [21], [29]-[32]. The common usage of those two tools in order to construct the joint general system reliability model related to its operation process [7]-[9], [11], [13]-[16], [23]-[25] and to apply it to the reliability analysis of complex technical systems is briefly presented and applied to the reliability evaluation of an exemplary complex system. The complex technical systems reliability improvement and decreasing the risk of exceeding a critical reliability state are of great value in the industrial practice [16], [18]-[19], [28]. There are needed the tools allowing for changing their operation processes after comparing the values of these characteristics with their values before their operation processes optimization in order to improve their reliability [5], [13]-[14], [16], [21], [27].

2. Complex system operation modeling
We assume that the system during its operation process is taking \( \nu, \nu \in \mathbb{N} \), different operation states \( z, z_1, \ldots, z_\nu \). Further, we define the system operation process \( Z(t), t \in <0, +\infty) \), with discrete operation states from the set \{ \( z, z_1, \ldots, z_\nu \) \}. Moreover, we assume that the system operation process \( Z(t) \) is a semi-Markov process [3], [12]-[13], [26] with the conditional sojourn times \( \theta_{bl} \) at the operation states \( z \), when its next operation state is \( z_l, b, l = 1,2,\ldots, \nu, b \neq l \). Under these assumptions, the system operation process may be described by:
- the vector \( [p_{b}(0)]_{\nu \times \nu} \) of the initial probabilities \( p_{b}(0) = P(Z(0) = z_{b}), b = 1,2,\ldots, \nu \), of the system operation process \( Z(t) \) staying at particular operation states at the moment \( t = 0 \);
- the matrix \( [p_{bl}]_{\nu \times \nu} \) of probabilities \( p_{bl}, b, l = 1,2,\ldots, \nu, b \neq l \), of the system operation process \( Z(t) \) transitions between the operation states \( z_b \) and \( z_l \);
- the matrix \([H_{i1}(t)]_{ov}\) of conditional distribution functions \(H_{i1}(t) = P(\theta_i < t), \quad t \geq 0, \quad b, l = 1, 2, \ldots, v, \quad b \neq l\), of the system operation process \(Z(t)\) conditional sojourn times \(\theta_i\) at the operation states. As the mean values \(E[\theta_i]\) of the conditional sojourn times \(\theta_i\) are given by 
\[
M_{bl} = E[\theta_i] = \int_0^t \theta_i dH_{i1}(t), \quad b, l = 1, 2, \ldots, v, \quad b \neq l, \quad (1)
\]
then from the formula for total probability, it follows that the unconditional distribution functions of the sojourn times \(\bar{\theta}_i, \quad b = 1, 2, \ldots, v\), of the system operation process \(Z(t)\) at the operation states \(z_s, \quad b = 1, 2, \ldots, v\), are given by [3], [16], [21], [26] 
\[
H_i(t) = \sum_{l=1}^{v} p_{bl} M_{bl}, \quad t \geq 0, \quad b = 1, 2, \ldots, v.
\]
Hence, the mean values \(E[\bar{\theta}_i]\) of the system operation process \(Z(t)\) unconditional sojourn times \(\bar{\theta}_i, \quad b = 1, 2, \ldots, v\), at the operation states are given by 
\[
M_b = E[\bar{\theta}_i] = \sum_{l=1}^{v} p_{bl} M_{bl}, \quad b = 1, 2, \ldots, v,
\]
where \(M_{bl}\) are defined by the formula (1). The limit values of the system operation process \(Z(t)\) transient probabilities at the particular operation states \(p_s(t) = P(Z(t) = z_s), \quad t \in <0, +\infty>, \quad b = 1, 2, \ldots, v\), are given by [3], [15], [21], [26]
\[
p_s = \lim_{t \to \infty} p_s(t) = \frac{\pi_b M_b}{\sum_{l=1}^{v} \pi_l M_l}, \quad b = 1, 2, \ldots, v, \quad (3)
\]
where \(M_b, \quad b = 1, 2, \ldots, v\), are given by (2), while the steady probabilities \(\pi_s\) of the vector \([\pi_s]_{sv}\) satisfy the system of equations
\[
\begin{align*}
[\pi_s] = [\pi_s][p_s] \\
\sum_{s=1}^{v} \pi_s = 1.
\end{align*} \quad (4)
\]
Other interesting characteristics of the system operation process \(Z(t)\) possible to obtain are its total sojourn times \(\hat{\theta}_b\) at the particular operation states \(\hat{\theta}_b, \quad b = 1, 2, \ldots, v\), during the fixed system operation time. It is well known [3], [16], [21], [26] that the system operation process total sojourn times \(\hat{\theta}_b\) at the particular operation states \(z_s, \quad b = 1, 2, \ldots, v\), for sufficiently large operation time \(\theta\), have approximately normal distributions with the expected value given by 
\[
\hat{M}_b = E[\hat{\theta}_b] = p_b \theta, \quad b = 1, 2, \ldots, v, \quad (5)
\]
where \(p_b\) are given by (3).

**Example**

We consider a series system \(S\) composed of the subsystems \(S_1\) and \(S_2\), with the scheme showed in Figure 1.

![Figure 1](image1.png)

**Figure 1.** The scheme of the exemplary system \(S\) reliability structure (6)

We assume that the subsystem \(S_1\) is a series-parallel system with the scheme given in Figure 2 and the subsystem \(S_2\) illustrated in Figure 3 is either a series-parallel system or a series-“2 out of 4” system.

![Figure 2](image2.png)

**Figure 2.** The scheme of the subsystem \(S_1\) reliability structure

![Figure 3](image3.png)

**Figure 3.** The scheme of the subsystem \(S_2\) reliability structure

The subsystems \(S_1\) and \(S_2\) are forming a general series reliability structure of the system presented in Figure 1. However, this system reliability structure and its subsystems and components reliability
depend on its changing in time operation states [13], [16], [26].
Under the assumption that the system operation conditions are changing in time, we arbitrarily fix the number of the system operation process states \( \nu = 4 \) and we distinguish the following as its operation states:
• an operation state \( z_i \) – the system is composed of the subsystem \( S_i \) with the scheme showed in Figure 2 that is a series-parallel system,
• an operation state \( z_2 \) – the system is composed of the system \( S_2 \) with the scheme showed in Figure 3 that is a series-parallel system,
• an operation state \( z_3 \) – the system is a series system with the scheme showed in Figure 1 composed of the subsystems \( S_1 \) and \( S_2 \) that are series-parallel systems with the schemes respectively given in Figure 2 and Figure 3,
• an operation state \( z_4 \) – the system is a series system with the scheme showed in Figure 1 composed of the subsystem \( S_1 \) and \( S_2 \), while the subsystem \( S_1 \) is a series-parallel system with the scheme given in Figure 2 and the subsystem \( S_2 \) is a series-“2 out of 4” system with the scheme given in Figure 3.

The influence of the above system operation states changing on the changes of the exemplary system reliability structure is indicated in these operation states above definitions and illustrated in Figures 1-3. Its influence on the system components reliability will be defined in this example continuation in Section 3.

We arbitrarily assume that the probabilities \( p_{z_i} \) of the exemplary system operation process transitions from operation state \( z_i \) into the operation state \( z_j \) are given in the matrix below

\[
[p_{z_i}] = \begin{bmatrix}
0 & 0.25 & 0.30 & 0.45 \\
0.20 & 0 & 0.25 & 0.55 \\
0.15 & 0.20 & 0 & 0.65 \\
0.40 & 0.25 & 0.35 & 0
\end{bmatrix}.
\]

We also arbitrarily fix the conditional mean values \( M_{z_1} = E[\theta_{z_1}] \), \( b, t = 1,2,3,4 \), of the exemplary system sojourn times at the particular operation states as follows:

\[
M_{z_1} = 190, \quad M_{z_2} = 480, \quad M_{z_3} = 200, \\
M_{z_4} = 320, \quad M_{z_5} = 510, \quad M_{z_6} = 440.
\]

This way, the exemplary system operation process is defined and we may find its main characteristics. Namely, applying (2), (6) and (7), the unconditional mean sojourn times at the particular operation states are:

\[
M_1 = 281.5, \quad M_2 = 73.0, \\
M_3 = 421.5, \quad M_4 = 409.5.
\]

Further, according to (4), after considering (6), we find the steady probabilities

\[
\pi_i = 0.216, \quad \pi_2 = 0.191, \\
\pi_3 = 0.237, \quad \pi_4 = 0.356.
\]

After considering the result (8) and (9), according to (3), the limit values of the exemplary system operation process transient probabilities \( p_{z_i}(t) \) at the operation states \( z_i \) are:

\[
p_1 = 0.190, \quad p_2 = 0.043, \\
p_3 = 0.312, \quad p_4 = 0.455.
\]

Hence, the expected values of the total sojourn times \( \hat{\theta}_b \), \( b = 1,2,3,4 \), of the exemplary system operation process at the particular operation states \( z_b \), \( b = 1,2,3,4 \), during the fixed operation time \( \theta = 1 \) year = 365 days, after applying (5), amount (in days):

\[
\hat{M}_1 = 69.3, \quad \hat{M}_2 = 15.7, \\
\hat{M}_3 = 113.9, \quad \hat{M}_4 = 166.1.
\]

### 3. Complex system reliability modeling

We assume that the changes of the operation states of the system operation process \( Z(t) \) have an influence on the system multistate components \( E_i \), \( i = 1,2,...,n \), reliability and the system reliability structure as well. Consequently, we denote the system multistate component \( E_i \), \( i = 1,2,...,n \), conditional lifetime in the reliability state subset \( [u,u+1,...,z] \) while the system is at the operation state \( z_b \), \( b = 1,2,...,v \), by \( T^{(z)}(u) \) and its conditional reliability function by the vector

\[
[R(t,z)]^{(z)} = [1, [R(t,1)]^{(z)}, ..., [R(t,z)]^{(z)}],
\]
with the coordinates defined by
for $t \in ]0, \infty[ $, $u = 1,2,\ldots, z $, $b = 1,2,\ldots, \nu $.

The reliability function $ [R(t,u)](13) $ is the conditional probability that the component $ i $ lifetime $ T_i^\nu(u) $ in the reliability state subset $ \{u,u+1,\ldots,z\} $ is greater than $ t $, while the system operation process $ Z(t) $ is at the operation state $ z_\nu $.

Similarly, we denote the system conditional lifetime in the reliability state subset $ \{u,u+1,\ldots,z\} $ while the system is at the operation state $ z_\nu $, $ b = 1,2,\ldots, \nu $, by $ T^\nu(u) $ and the conditional reliability function of the system by the vector

$$ [R(t,\cdot)](13) = [1, [R(t,1)](13), \ldots, [R(t,z)](13)], $$

with the coordinates defined by

$$ [R(t,u)](13) = \mathbb{P}(T_i^\nu(u) > t | Z(t) = z_\nu) \tag{12} $$

for $ t \in ]0, \infty[ $, $ u = 1,2,\ldots, z $, $ b = 1,2,\ldots, \nu $.

The reliability function $ [R(t,u)](13) $ is the conditional probability that the system lifetime $ T^\nu(u) $ in the reliability state subset $ \{u,u+1,\ldots,z\} $ is greater than $ t $, while the system operation process $ Z(t) $ is at the operation state $ z_\nu $.

Further, we denote the system unconditional lifetime in the reliability state subset $ \{u,u+1,\ldots,z\} $ by $ T(u) $ and the unconditional reliability function of the system by the vector

$$ R(t,\cdot) = [1, R(t,1), \ldots, R(t,z)], $$

with the coordinates defined by

$$ R(t,u) = \mathbb{P}(T(u) > t) \tag{13} $$

for $ t \in ]0, \infty[ $, $ u = 1,2,\ldots, z $.

In the case when the system operation time $ \theta $ is large enough, the coordinates of the unconditional reliability function of the system defined by (13) are given by

$$ R(t,u) = \sum_{i=1}^\nu p_i [R(t,u)](13) \text{ for } t \geq 0, u = 1,2,\ldots, z, \tag{14} $$

where $ [R(t,u)](13), u = 1,2,\ldots, z, b = 1,2,\ldots, \nu $, are the coordinates of the system conditional reliability functions defined by (12) and $ p_i, b = 1,2,\ldots, \nu $, are the system operation process limit transient probabilities given by (3).

Thus, the mean value $ \mu(u) = \mathbb{E}[T(u)] $ of the system unconditional lifetime $ T(u) $ in the reliability state subset $ \{u,u+1,\ldots,z\} $ is given by [16], [26]

$$ \mu(u) = \sum_{b=1}^\nu p_b \mu_b(u), \quad u = 1,2,\ldots, z. \tag{15} $$

where $ \mu_b(u) = \mathbb{E}[T^\nu(u)] $ are the mean values of the system conditional lifetimes $ T^\nu(u) $ in the reliability state subset $ \{u,u+1,\ldots,z\} $ at the operation state $ z_b $, $ b = 1,2,\ldots, \nu $, given by

$$ \mu_b(u) = \int_0^\infty [R(t,u)](13) dt, \quad u = 1,2,\ldots, \nu. \tag{16} $$

$$ [R(t,u)](13), \quad u = 1,2,\ldots, z, \quad b = 1,2,\ldots, \nu, \text{ are defined by (12) and } p_i \text{ are given by (3). Since the relationships between the system unconditional lifetimes } T(u) \text{ in the particular reliability states and the system unconditional lifetimes } T^\nu(u) \text{ in the reliability state subsets can be expressed by}

$$ \overline{T}(u) = T(u) - T(u+1), \quad u = 0,1,\ldots, z, $$

$$ \overline{T}(z) = T(z), $$

then we get the following formulae for the mean values of the unconditional lifetimes of the system in particular reliability states

$$ \overline{\mu}(u) = \mu(u) - \mu(u+1), \quad u = 0,1,\ldots, z-1, $$

$$ \overline{\mu}(z) = \mu(z), \tag{17} $$

where $ \mu(u), \quad u = 0,1,\ldots, z, \text{ are given by (15). Moreover, if } s(t) \text{ is the system reliability state at the moment } t, \quad t \in ]0, \infty[, \text{ and } r, r \in \{1,2,\ldots, z\}, \text{ is the system critical reliability state, then the system risk function}

$$ r(t) = \mathbb{P}(s(t) < r | s(0) = z) = \mathbb{P}(T(r) \leq t), \quad t \in ]0, \infty[, $$

defined as the probability that the system is in the subset of states worse than the critical state $ r, r \in \{1,\ldots, z\} \text{ while it was in the state } z \text{ at the moment } t = 0 $ is given by [16]

$$ r(t) = 1 - R(t,r), \quad t \in ]0, \infty[, \tag{18} $$

where $ R(t,r) $ is the coordinate of the system unconditional reliability function given by (14) for
\( u = r \) and if \( \tau \) is the moment when the system risk function exceeds a permitted level \( \delta \), then
\[
\tau = r^{-1}(\delta),
\]
where \( r^{-1}(t) \), if it exists, is the inverse function of the risk function \( r(t) \) given by (18).

**Example** (continuation)
In Section 2, it is fixed that the exemplary system reliability structure and its subsystems and components reliability depend on its changing in time operation states. Considering the assumptions and agreements of these sections, we assume that its subsystems \( S_j \), \( \nu = 1,2 \), are composed of four-state, i.e. \( z = 3 \), components \( E^{(\nu)}_b \), \( \nu = 1,2 \), having the conditional reliability functions given by the vector
\[
[R^{(\nu)}_b(t,\cdot)]^{(i)} = [1, [R^{(\nu)}_b(t,1)]^{(i)}, [R^{(\nu)}_b(t,2)]^{(i)}, [R^{(\nu)}_b(t,3)]^{(i)}], \quad b = 1,2,3,4,
\]
with the exponential co-ordinates
\[
[R^{(\nu)}_b(t,1)]^{(i)} = \exp[-\lambda^{(\nu)}_b(1)]^{(i)}),
[R^{(\nu)}_b(t,2)]^{(i)} = \exp[-\lambda^{(\nu)}_b(2)]^{(i)}),
[R^{(\nu)}_b(t,3)]^{(i)} = \exp[-\lambda^{(\nu)}_b(3)]^{(i)}),
\]
different at various operation states \( z_i \), \( b = 1,2,3,4 \), and with the intensities of departure from the reliability state subsets \{1,2,3\}, \{2,3\}, \{3\}, respectively
\[
[\lambda^{(\nu)}_b(1)]^{(i)}), [\lambda^{(\nu)}_b(2)]^{(i)}), [\lambda^{(\nu)}_b(3)]^{(i)}), \quad b = 1,2,3,4.
\]
The influence of the system operation states changing on the changes of the system reliability structure and its components reliability functions is as follows.

At the system operation state \( z_i \), the system is composed of the series-parallel subsystem \( S_i \) with the structure showed in Figure 2, containing two identical series subsystems \( (k^{(1)} = 2) \), each composed of sixty components \( (l^{(1)} = 60, l^{(2)} = 60) \) with the exponential reliability functions. In both series subsystems of the subsystem \( S_i \) there are respectively:
- the components \( E^{(1)}_b \), \( i = 1,2 \), \( j = 1,2,...,40 \), with the conditional reliability function coordinates
\[
[R^{(1)}_b(t,1)]^{(i)} = \exp[-0.0008r],
\]
- the components \( E^{(2)}_b \), \( i = 1,2,3,4 \), \( j = 1,2,...,40 \), with the conditional reliability function co-ordinates
\[
[R^{(2)}_b(t,2)]^{(i)} = \exp[-0.0009r],
[R^{(2)}_b(t,3)]^{(i)} = \exp[-0.0010r], \quad i = 1,2, \quad j = 1,2,...,40;\]
- the components \( E^{(3)}_b \), \( i = 1,2 \), \( j = 41,42,...,60 \), with the conditional reliability function coordinates
\[
[R^{(3)}_b(t,1)]^{(i)} = \exp[-0.0011t],
[R^{(3)}_b(t,2)]^{(i)} = \exp[-0.0012t],
[R^{(3)}_b(t,3)]^{(i)} = \exp[-0.0013t], \quad i = 1,2, \quad j = 41,42,...,60.
\]
Thus, at the operational state \( z_i \), the system is identical with the subsystem \( S_i \) that is a four-state series-parallel system with its structure shape parameters \( l^{(1)} = 60, l^{(2)} = 60 \), and according to Proposition 1 given in [17], its conditional reliability function is given by
\[
[R(t,\cdot)]^{(i)} = [1, [R(t,1)]^{(i)}, [R(t,2)]^{(i)}, [R(t,3)]^{(i)}] \quad (20)
\]
for \( t \geq 0 \), where
\[
[R(t,1)]^{(i)} = 2\exp[-0.054t] - \exp[-0.108t],
[R(t,2)]^{(i)} = 2\exp[-0.060t] - \exp[-0.120t],
[R(t,3)]^{(i)} = 2\exp[-0.066t] - \exp[-0.132t]. \quad (21)
\]
The expected values and standard deviations of the system conditional lifetimes in the reliability state subsets \{1,2,3\}, \{2,3\}, \{3\} at the operation state \( z_i \), calculated from the results given by (21), according to (16), respectively are:
\[
\mu_1 = \int_0^\infty [R(t,1)]^{(i)} dt = 27.78,
\mu_1 = \int_0^\infty [R(t,2)]^{(i)} dt = 25.00,
\mu_1 = \int_0^\infty [R(t,3)]^{(i)} dt = 22.73. \quad (22)
\]
At the system operation state \( z_i \), the system is composed of the series-parallel subsystem \( S_i \) with the structure showed in Figure 3, containing four identical series subsystems \( (k^{(3)} = 4) \), each composed of eighty components \( (l^{(1)} = 80, l^{(2)} = 80, l^{(3)} = 80, l^{(4)} = 80) \) with the exponential reliability functions. In all series subsystems of the subsystem \( S_i \) there are respectively:
- the components \( E^{(3)}_b \), \( i = 1,2,3,4 \), \( j = 1,2,...,40 \), with the conditional reliability function co-ordinates
\[ R^{(2)}(t,1)^{(2)} = \exp[-0.0012t], \]
\[ R^{(2)}(t,2)^{(2)} = \exp[-0.0014t], \]
\[ R^{(2)}(t,3)^{(2)} = \exp[-0.0016t], \]
\( j = 1,2,\ldots,40; \)

- the components \( E_{\text{i}}^{(2)}, \ i = 1,2,3,4, \ j = 21,22,\ldots,40, \)

with the conditional reliability function co-ordinates

\[ R^{(2)}(t,1)^{(3)} = \exp[-0.0018t], \]
\[ R^{(2)}(t,2)^{(3)} = \exp[-0.0020t], \]
\[ R^{(2)}(t,3)^{(3)} = \exp[-0.0022t], \]
\( j = 41,42,\ldots,80. \)

Thus, at the operation state \( z_{\text{A}} \), the system is identical with the subsystem \( S_{\text{A}} \) that is a four-state series-parallel system with its structure shape parameters \( k^{(1)} = 4, \ l_{1}^{(2)} = 80, \ l_{2}^{(2)} = 80, \ l_{3}^{(2)} = 80 \), and according to Proposition 1 given in [17], its conditional reliability function is given by

\[ R(t,\cdot)^{(2)} = [1, R(t,1)^{(2)}, R(t,2)^{(2)}, R(t,3)^{(2})] \] (23)

for \( t \geq 0 \), where

\[ R(t,1)^{(2)} = 4 \exp[-0.128t] - 6 \exp[-0.256t] + 4 \exp[-0.384t] - \exp[-0.512t], \]
\[ R(t,2)^{(2)} = 4 \exp[-0.140t] - 6 \exp[-0.280t] + 4 \exp[-0.420t] - \exp[-0.560t], \]
\[ R(t,3)^{(2)} = 4 \exp[-0.152t] - 6 \exp[-0.304t] + 4 \exp[-0.456t] - \exp[-0.608t]. \] (24)

The expected values and standard deviations of the system conditional lifetimes in the reliability state subsets \{1,2,3\}, \{2,3\}, \{3\} at the operation state \( z_{\text{A}} \), calculated from the results given by (24), according to (16), respectively are:

\[ \mu_{1}(1) = \int_{0}^{\infty} R(t,1)^{(2)} dt \cong 16.27, \]
\[ \mu_{2}(2) = \int_{0}^{\infty} R(t,2)^{(2)} dt \cong 14.88, \]
\[ \mu_{3}(3) = \int_{0}^{\infty} R(t,3)^{(2)} dt \cong 13.71. \] (25)

At the operation state \( z_{\text{A}} \), the system is a series system with the structure showed in Figure 1, composed of two series-parallel subsystems \( S_{\text{A}} \) and \( S_{\text{B}} \) illustrated respectively in Figure 2 and Figure 3. The subsystem \( S_{\text{A}} \) with the structure showed in Figure 2, consists of two identical series subsystems \( (k^{(1)} = 2) \), each composed of sixty components \( (l_{1}^{(2)} = 60, l_{2}^{(2)} = 60) \) with the exponential reliability functions. In both series subsystems of the subsystem \( S_{\text{A}} \) there are respectively:

- the components \( E_{\text{i}}^{(2)}, \ i = 1,2, j = 1,2,\ldots,40, \)

with the conditional reliability function co-ordinates

\[ R^{(2)}(t,1)^{(2)} = \exp[-0.0009t], \]
\[ R^{(2)}(t,2)^{(2)} = \exp[-0.0010t], \]
\[ R^{(2)}(t,3)^{(2)} = \exp[-0.0011t], \]
\( i = 1,2, j = 1,2,\ldots,40; \)

- the components \( E_{\text{i}}^{(2)}, \ i = 41,42,\ldots,60, \)

with the conditional reliability function co-ordinates

\[ R^{(2)}(t,1)^{(2)} = \exp[-0.0012t], \]
\[ R^{(2)}(t,2)^{(2)} = \exp[-0.0014t], \]
\[ R^{(2)}(t,3)^{(2)} = \exp[-0.0016t], \]
\( i = 1,2, j = 41,42,\ldots,60. \)

Thus, at the operation state \( z_{\text{A}} \), the subsystem \( S_{\text{A}} \) is a four-state series-parallel system with its structure shape parameters \( k^{(1)} = 2, \ l_{1}^{(2)} = 60, \ l_{2}^{(2)} = 60 \), and according to Proposition 1 given in [17], its conditional reliability function is given by

\[ R^{(2)}(t,\cdot)^{(3)} = [1, R^{(2)}(t,1)^{(3)}, R^{(2)}(t,2)^{(3)}, R^{(2)}(t,3)^{(3})] \] (26)

for \( t \geq 0 \), where

\[ R^{(2)}(t,1)^{(3)} = 2 \exp[-0.060t] - \exp[-0.120t], \]
\[ R^{(2)}(t,2)^{(3)} = 2 \exp[-0.068t] - \exp[-0.136t], \]
\[ R^{(2)}(t,3)^{(3)} = 2 \exp[-0.076t] - \exp[-0.152t]. \] (27)

The subsystem \( S_{\text{B}} \) with the structure showed in Figure 3, consists of four identical series subsystems \( (k^{(1)} = 4) \), each composed of eighty components \( (l_{1}^{(2)} = 80, l_{2}^{(2)} = 80, l_{3}^{(2)} = 80, l_{4}^{(2)} = 80) \) with the exponential reliability functions given below. In all series subsystems of the subsystem \( S_{\text{B}} \) there are respectively:

- the components \( E_{\text{i}}^{(3)}, \ i = 1,2,3,4, \ j = 1,2,\ldots,40, \)

with the conditional reliability function co-ordinates

\[ R^{(3)}(t,1)^{(3)} = \exp[-0.0009t], \]
\[ R^{(3)}(t,2)^{(3)} = \exp[-0.0011t], \]
\[ R^{(3)}(t,3)^{(3)} = \exp[-0.0012t], \]
\( i = 1,2,3,4, j = 1,2,\ldots,40; \)
- the components \( E^{(i)}_j \), \( i = 1, 2, 3, 4, \ j = 41, 42, \ldots, 80 \), with the conditional reliability function co-ordinates

\[
\begin{align*}
[R^{(0)}_0(t.1)]^{(j)} &= \exp[-0.0014t] , \\
[R^{(2)}_0(t.2)]^{(j)} &= \exp[-0.0016t] , \\
[R^{(2)}_0(t.3)]^{(j)} &= \exp[-0.0018t], \ i = 1, 2, 3, 4, \ j = 41, 42, \ldots, 80.
\end{align*}
\]

Thus, at the operation state \( z_i \), the subsystem \( S_i \) is a four-state series-parallel system with its structure shape parameters \( k^{(0)} = 4 \), \( l^{(0)}_1 = 80 \), \( l^{(0)}_2 = 80 \), \( l^{(1)}_1 = 80 \), \( l^{(1)}_2 = 80 \) and, according to Proposition 1 given in [17], its conditional reliability function is given by

\[
[R^{(0)}(t.,.)]^{(i)} = [1, \ [R^{(0)}(t,1)]^{(i)}, \ [R^{(0)}(t,2)]^{(i)}, \\
[R^{(0)}(t,3)]^{(i)}], \ t \geq 0, \tag{28}
\]

where

\[
\begin{align*}
[R^{(0)}(t,1)]^{(i)} &= 4 \exp[-0.096t] - 6 \exp[-0.192t] + 4 \exp[-0.288t] - \exp[-0.384t], \\
[R^{(0)}(t,2)]^{(i)} &= 4 \exp[-0.108t] - 6 \exp[-0.216t] + 4 \exp[-0.324t] - \exp[-0.432t], \\
[R^{(0)}(t,3)]^{(i)} &= 4 \exp[-0.120t] - 6 \exp[-0.240t] + 4 \exp[-0.360t] - \exp[-0.480t]. \tag{29}
\end{align*}
\]

Considering that the system at the operation state \( z_i \) is a four-state series system composed of subsystems \( S_i \) and \( S_j \), after applying the formulae appearing after Definition 3.4 in [16] and (27) and (29), its conditional reliability function is given by

\[
[R(t,.)]^{(i)} = [1, [R(t,1)]^{(i)}, [R(t,2)]^{(i)}, [R(t,3)]^{(i)}] \tag{30}
\]

for \( t \geq 0 \), where

\[
\begin{align*}
[R(t,1)]^{(i)} &= 8 \exp[-0.156t] - 12 \exp[-0.252t] + 8 \exp[-0.348t] - 2 \exp[-0.424t] - 4 \exp[-0.216t] + 6 \exp[-0.312t] - 4 \exp[-0.408t] + \exp[-0.504t], \\
[R(t,2)]^{(i)} &= 8 \exp[-0.176t] - 12 \exp[-0.284t] + 8 \exp[-0.392t] - 2 \exp[-0.500t] - 4 \exp[-0.236t] + 6 \exp[-0.344t] - 4 \exp[-0.452t] + \exp[-0.560t], \\
[R(t,3)]^{(i)} &= 8 \exp[-0.196t] - 12 \exp[-0.316t] + 8 \exp[-0.436t] - 2 \exp[-0.556t] - 4 \exp[-0.256t] + 6 \exp[-0.376t] - 4 \exp[-0.496t] + \exp[-0.616t]. \tag{31}
\end{align*}
\]

The expected values and standard deviations of the system conditional lifetimes in the reliability state subsets \( \{1, 2, 3\}, \{2, 3\}, \{3\} \) at the operation state \( z_i \), calculated from the results given by (31), according to (16), respectively are:

\[
\begin{align*}
\mu_i(1) &= \int_0^\infty [R(t,1)]^{(3)} dt \equiv 14.82, \\
\mu_i(2) &= \int_0^\infty [R(t,2)]^{(3)} dt \equiv 13.04, \\
\mu_i(3) &= \int_0^\infty [R(t,3)]^{(3)} dt \equiv 11.48. \tag{32}
\end{align*}
\]

At the system operation state \( z_i \), the system is a series system with the scheme showed in Figure 1, composed of the subsystems \( S_i \) and \( S_j \), illustrated respectively in Figure 2 and Figure 3, whereas the subsystem \( S_i \) is a series-parallel system and the subsystem \( S_j \) is a series-“2 out of 4” system.

The subsystem \( S_i \) consists of two identical series subsystems \( (k^{(i)} = 2) \), each composed of sixty components \( (l^{(i)} = 60, l^{(i)} = 60) \) with the exponential reliability functions the same as at the operation state \( z_i \). Thus, according to (21), the subsystem \( S_i \) conditional reliability function at the operation state \( z_i \), is given by

\[
[R^{(i)}(t,.)]^{(i)} = [1, [R^{(i)}(t,1)]^{(i)}, [R^{(i)}(t,2)]^{(i)}, \\
[R^{(i)}(t,3)]^{(i)}] \tag{33}
\]

for \( t \geq 0 \), where

\[
\begin{align*}
[R^{(i)}(t,1)]^{(i)} &= 2 \exp[-0.054t] - \exp[-0.108t], \\
[R^{(i)}(t,2)]^{(i)} &= 2 \exp[-0.060t] - \exp[-0.120t], \\
[R^{(i)}(t,3)]^{(i)} &= 2 \exp[-0.066t] - \exp[-0.132t]. \tag{34}
\end{align*}
\]

The subsystem \( S_j \) consists of four identical series subsystems \( (k^{(j)} = 4) \), each composed of eighty components \( (l^{(j)} = 80, l^{(j)} = 80, l^{(j)} = 80, l^{(j)} = 80) \) with the exponential reliability functions the same as at the operation state \( z_j \) and is a series-“2 out of 4” system \( (m = 2) \). Thus, at the operation state \( z_j \), the subsystem \( S_j \) is a four-state series-“2 out of 4” system, with its structure shape parameters \( k^{(j)} = 4 \), \( l^{(j)} = 80 \), \( l^{(j)} = 80, l^{(j)} = 80, l^{(j)} = 80 \), and according to Proposition 1 given in [17], its conditional reliability function is given by

\[
[R^{(j)}(t,.)]^{(j)} = [1, [R^{(j)}(t,1)]^{(j)}, [R^{(j)}(t,2)]^{(j)}, \\
[R^{(j)}(t,3)]^{(j)}] \tag{35}
\]
for $t \geq 0$, where
\[
\begin{align*}
[R(t,1)]^{(4)} &= 6 \exp[-0.256t] - 8 \exp[-0.384t] \\
+ 3 \exp[-0.512t], \\
[R(t,2)]^{(4)} &= 6 \exp[-0.280t] - 8 \exp[-0.420t] \\
+ 3 \exp[-0.560t], \\
[R(t,3)]^{(4)} &= 6 \exp[-0.304t] - 8 \exp[-0.456t] \\
+ 3 \exp[-0.608t].
\end{align*}
\] (36)

Considering that the system at the operation state $z_i$ is a four-state series system composed of subsystems $S_1$ and $S_2$, after applying the formulae appearing after Definition 3.4 in [16] and (34) and (36), its conditional reliability function is given by
\[
[R(t,\cdot)]^{(4)} = [I, [R(t,1)]^{(4)}, [R(t,2)]^{(4)}, [R(t,3)]^{(4)}] 
\] (37)

for $t \geq 0$, where
\[
\begin{align*}
[R(t,1)]^{(4)} &= 12 \exp[-0.310t] - 6 \exp[-0.364t] \\
- 16 \exp[-0.438t] + 8 \exp[-0.492t] \\
+ 6 \exp[-0.566t] - 3 \exp[-0.620t], \\
[R(t,2)]^{(4)} &= 12 \exp[-0.340t] - 6 \exp[-0.400t] \\
- 16 \exp[-0.480t] + 8 \exp[-0.540t] \\
+ 6 \exp[-0.620t] - 3 \exp[-0.680t], \\
[R(t,3)]^{(4)} &= 12 \exp[-0.370t] - 6 \exp[-0.436t] \\
- 16 \exp[-0.522t] + 8 \exp[-0.588t] \\
+ 6 \exp[-0.674t] - 3 \exp[-0.740t].
\end{align*}
\] (38)

The mean values of the system sojourn times $T(u)$ in the reliability state subsets after applying the formula (38) and (16), are:
\[
\begin{align*}
\mu_1(1) &= \int_0^\infty [R(t,1)]^{(4)} dt \equiv 7.72, \\
\mu_2(2) &= \int_0^\infty [R(t,2)]^{(4)} dt \equiv 7.04, \\
\mu_3(3) &= \int_0^\infty [R(t,3)]^{(4)} dt \equiv 6.47.
\end{align*}
\] (39)

In the case when the system operation time is large enough its unconditional four-state reliability function is given by the vector
\[
R(t,\cdot) = [I, R(t,1), R(t,2), R(t,3)] 
\] for $t \geq 0$. (40)

where according to (14) and considering the exemplary system operation process transient probabilities at the operation states determined by (10), the vector co-ordinates are given respectively by
\[
R(t,1) = p_1[R(t,1)]^{(4)} + p_2[R(t,1)]^{(4)} + p_3[R(t,1)]^{(4)} \\
+ p_4[R(t,1)]^{(4)} \\
= 0.190 \cdot [R(t,1)]^{(4)} + 0.043 \cdot [R(t,1)]^{(4)} \\
+ 0.312 \cdot [R(t,1)]^{(4)} + 0.455 \cdot [R(t,1)]^{(4)},
\]
\[
R(t,2) = p_1[R(t,2)]^{(4)} + p_2[R(t,2)]^{(4)} + p_3[R(t,2)]^{(4)} \\
+ p_4[R(t,2)]^{(4)} \\
= 0.190 \cdot [R(t,2)]^{(4)} + 0.043 \cdot [R(t,2)]^{(4)} \\
+ 0.312 \cdot [R(t,2)]^{(4)} + 0.455 \cdot [R(t,2)]^{(4)},
\]
\[
R(t,3) = p_1[R(t,3)]^{(4)} + p_2[R(t,3)]^{(4)} \\
+ p_3[R(t,3)]^{(4)} \\
= 0.190 \cdot [R(t,3)]^{(4)} + 0.043 \cdot [R(t,3)]^{(4)} \\
+ 0.312 \cdot [R(t,3)]^{(4)} + 0.455 \cdot [R(t,3)]^{(4)}.
\] (41)

The graph of the four-state exemplary system reliability function is illustrated in Figure 4.

![Figure 4. The graph of the exemplary system reliability function $R(t,\cdot)$ coordinates](image)

The expected values of the system unconditional lifetimes in the reliability state subsets $\{1,2,3\}$, $\{2,3\}$, $\{3\}$, calculated from the results given by (41) according to (16) and considering (15) and (21), (24), (31), (38), respectively are:
\[
\begin{align*}
\mu(1) &= p_1\mu_1(1) + p_2\mu_2(1) + p_3\mu_3(1) + p_4\mu_4(1) \\
&= 0.190 \cdot 27.78 + 0.043 \cdot 16.27 + 0.312 \cdot 14.82 \\
&+ 0.455 \cdot 7.72 \equiv 14.11,
\]
\[
\mu(2) &= p_1\mu_1(2) + p_2\mu_2(2) + p_3\mu_3(2) \\
&+ p_4\mu_4(2) \\
&= 0.190 \cdot 25.00 + 0.043 \cdot 14.88 + 0.312 \cdot 13.04
\]
Further, considering (17) and (42), the mean values of the system unconditional lifetimes in the particular reliability states 1, 2, 3, respectively are:

\[
\overline{\mu}(1) = \mu(1) - \mu(2) = 1.45,
\overline{\mu}(2) = \mu(2) - \mu(3) = 1.23,
\overline{\mu}(3) = \mu(3) = 11.43.
\]

(43)

Since the critical reliability state is \( r = 2 \), then the system risk function, according to (18), is given by

\[
r(t) = 1 - R(t, 2) \text{ for } t \geq 0,
\]

(44)

where \( R(t, 2) \) is given by (41).

Hence, by (19), the moment when the system risk function exceeds a permitted level, for instance \( \delta = 0.05 \), is

\[
\tau = r^{-1}(\delta) = 2.25.
\]

(45)

The graph of the risk function \( r(t) \) of the exemplary four-state system operating at the variable conditions is given in Figure 5.

![Figure 5. The graph of the exemplary system risk function \( r(t) \)](image)

### 4. Complex system reliability and operation optimization

Considering the equation (14), it is natural to assume that the system operation process has a significant influence on the system reliability. This influence is also clearly expressed in the equation (15) for the mean values of the system unconditional lifetimes in the reliability state subsets.

From the linear equation (15), we can see that the mean value of the system unconditional lifetime \( \mu(u), u = 1, 2, \ldots, z \), is determined by the limit values of transient probabilities \( p_b, b = 1, 2, \ldots, v \), of the system operation process at the operation states given by (3) and the mean values \( \mu_s(u), b = 1, 2, \ldots, v, u = 1, 2, \ldots, z \), of the system conditional lifetimes in the reliability state subsets \( \{u, u+1, \ldots, z\}, u = 1, 2, \ldots, z \), given by (16). Therefore, the system lifetime optimization approach based on the linear programming [13], [15], [17], [22], can be proposed. Namely, we may look for the corresponding optimal values \( \hat{p}_b, b = 1, 2, \ldots, v \), of the transient probabilities \( p_b, b = 1, 2, \ldots, v \), of the system operation process at the operation states to maximize the mean value \( \mu(u) \) of the unconditional system lifetimes in the reliability state subsets \( \{u, u+1, \ldots, z\}, u = 1, 2, \ldots, z \), under the assumption that the mean values \( \mu_s(u), b = 1, 2, \ldots, v, u = 1, 2, \ldots, z \), of the system conditional lifetimes in the reliability state subsets are fixed. As a special and practically important case of the above formulated system lifetime optimization problem, if \( r, r = 1, 2, \ldots, z \), is a system critical reliability state, we may look for the optimal values \( \hat{p}_b, b = 1, 2, \ldots, v \), of the transient probabilities \( p_b, b = 1, 2, \ldots, v \), of the system operation process at the system operation states to maximize the mean value \( \mu(r) \) of the unconditional system lifetime in the reliability state subset \( \{r, r+1, \ldots, z\}, r = 1, 2, \ldots, z \), under the assumption that the mean values \( \mu_s(r), b = 1, 2, \ldots, v, r = 1, 2, \ldots, z \), of the system conditional lifetimes in this reliability state subset are fixed. More exactly, we may formulate the optimization problem as a linear programming model with the objective function of the following form

\[
\mu(r) = \sum_{b=1}^{v} \hat{p}_b \mu_s(r)
\]

(46)

for a fixed \( r \in \{1, 2, \ldots, z\} \) and with the following bound constraints

\[
\hat{p}_b \leq p_b \leq \bar{p}_b, b = 1, 2, \ldots, v, \sum_{b=1}^{v} p_b = 1,
\]

(47)

where \( \mu_s(r) \), \( \mu_s(r) \geq 0, b = 1, 2, \ldots, v \), are fixed mean values of the system conditional lifetimes in the reliability state subset \( \{r, r+1, \ldots, z\} \) and

\[
\hat{p}_b, 0 \leq \hat{p}_b \leq 1 \text{ and } \bar{p}_b, 0 \leq \bar{p}_b \leq 1, \hat{p}_b \leq \bar{p}_b,
\]

(48)
are lower and upper bounds of the unknown transient probabilities \( p_i, \ b = 1,2,...,\nu, \) respectively.

Now, we can obtain the optimal solution of the formulated by (46)-(48) the linear programming problem, i.e. we can find the optimal values \( \hat{p}_i \) of the transient probabilities \( p_i, \ b = 1,2,...,\nu, \) that maximize the objective function given by (1).

First, we arrange the system conditional lifetime mean values \( \mu_i(r), \ b = 1,2,...,\nu, \) in non-increasing order \( \mu_i(r) \geq \mu_{i_2}(r) \geq \ldots \geq \mu_{i_{\nu}}(r), \) where \( b \in \{1,2,...,\nu\} \) for \( i = 1,2,...,\nu. \)

Next, we substitute

\[
x_i = p_i, \ \hat{x}_i = \hat{p}_i, \ \hat{x}_i = \hat{p}_i \text{ for } i = 1,2,...,\nu
\] (49)

and we maximize with respect to \( x_i, \ i = 1,2,...,\nu, \) the linear form (46) that after this transformation takes the form

\[
\mu(r) = \sum_{i=1}^{\nu} x_i \mu_i(r)
\] (50)

for a fixed \( r \in \{1,2,...,\nu\} \) with the following bound constraints

\[
\hat{x}_i \leq x_i \leq \check{x}_i, \ i = 1,2,...,\nu, \ \check{x}_i, \ i = 1,2,...,\nu,
\] (51)

where \( \mu_i(r), \ \mu_j(r) \geq 0, \ i = 1,2,...,\nu, \) are fixed mean values of the system conditional lifetimes in the reliability state subset \( \{r, r+1,...,\nu\} \) arranged in non-increasing order and

\[
\hat{x}_i, \ 0 \leq \hat{x}_i \leq 1 \text{ and } \check{x}_i, \ 0 \leq \check{x}_i \leq 1, \ \hat{x}_i \leq \check{x}_i, \ i = 1,2,...,\nu, \n\] (52)

are lower and upper bounds of the unknown probabilities \( x_i, \ i = 1,2,...,\nu, \) respectively.

To find the optimal values of \( x_i, \ i = 1,2,...,\nu, \) we define

\[
\check{x}_i = \sum_{r=1}^{\nu} \hat{x}_i, \ \hat{y}_i = 1 - \check{x}_i
\] (53)

and

\[
\check{x}_i = 0, \ \check{x}_i = 0 \text{ and } \hat{y}_i = \sum_{r=1}^{\nu} \hat{x}_i, \ \hat{x}_i = \sum_{r=1}^{\nu} \hat{x}_i
\] (54)

for \( i = 1,2,...,\nu. \)

Next, we find the largest value \( I \in \{0,1,...,\nu\} \) such that

\[
\hat{x}_i - \check{x}_i < \hat{y}_i \quad (55)
\]

and we fix the optimal solution that maximize (50) in the following way:

i) if \( I = 0, \) the optimal solution is

\[
\hat{x}_i = \hat{y}_i + \check{x}_i \text{ and } \check{x}_i = \check{x}_i \text{ for } i = 2,3,...,\nu;
\] (56)

ii) if \( 0 < I < \nu, \) the optimal solution is

\[
\hat{x}_i = \hat{x}_i \text{ for } i = 1,2,...,I, \ \hat{x}_{I+1} = \hat{y}_i - \hat{x}_i + \check{x}_i \text{ and } \check{x}_i = \check{x}_i \text{ for } i = I + 2, I + 3,...,\nu;
\] (57)

iii) if \( I = \nu, \) the optimal solution is

\[
\hat{x}_i = \check{x}_i \text{ for } i = 1,2,...,\nu.
\] (58)

Finally, after making the inverse to (49) substitution, we get the optimal limit transient probabilities

\[
\hat{p}_i = \hat{x}_i \text{ for } i = 1,2,...,\nu,
\] (59)

that maximize the system mean lifetime in the reliability state subset \( \{r, r+1,...,\nu\}, \) defined by the linear form (46), giving its maximum value in the following form

\[
\mu(r) = \sum_{i=1}^{\nu} \hat{p}_i \mu_i(r)
\] (60)

for a fixed \( r \in \{1,2,...,\nu\}. \)

From the expression (60) for the maximum mean value \( \hat{\mu}(r) \) of the system unconditional lifetime in the reliability state subset \( \{r, r+1,...,\nu\}, \) replacing in it the critical reliability state \( r \) by the reliability state \( u, \ u = 1,2,...,\nu, \) we obtain the corresponding optimal solutions for the mean values of the system unconditional lifetimes in the reliability state subsets \( \{u, u+1,...,\nu\} \) of the form

\[
\hat{\mu}(u) = \sum_{i=1}^{\nu} \hat{p}_i \mu_i(u) \text{ for } u = 1,2,...,\nu.
\] (61)

Further, according to (13)-(14), the corresponding optimal unconditional multistate reliability function of the system is the vector

\[
\hat{R}(t) = [1, \hat{R}(t,1), ..., \hat{R}(t,z)].
\] (62)
with the coordinates given by

\[ \dot{R}(t,u) \equiv \sum_{z=0}^{u} \hat{p}_{z} [R(t,u)]^{(z)} \quad \text{for} \quad t \geq 0, \quad u = 1, 2, \ldots, z. \]  \hfill (63)

And, by (17), the optimal solutions for the mean values of the system unconditional lifetimes in the particular reliability states are

\[ \tilde{\mu}(u) = \mu(u) - \mu(u+1), \quad u = 1, \ldots, z-1, \]
\[ \tilde{\mu}(z) = \mu(z). \]  \hfill (64)

Moreover, considering (18) and (19), the corresponding optimal system risk function and the optimal moment when the risk exceeds a permitted level \( \delta \) respectively are given by

\[ \dot{r}(t) = 1 - \dot{R}(t,r), \quad t \geq 0, \]  \hfill (65)

and

\[ \dot{\tau} = \dot{r}^{-1}(\delta), \]  \hfill (66)

where \( \dot{R}(t,r) \) is given by (63) for \( u = r \) and \( \dot{r}^{-1}(t) \), if it exists, is the inverse function of the optimal risk function \( \dot{r}(t) \).

Replacing in (3) the limit transient probabilities \( p_{s} \) of the system operation process at the operation states by their optimal values \( \hat{p}_{s} \), maximizing the mean value \( \mu(r) \) of the system lifetime in the reliability states subset \{r, r+1, \ldots, z\} defined by (46) and the mean values \( M_{b} \) of the unconditional sojourn times at the operation states by their corresponding unknown optimal values \( \hat{M}_{b} \), we get the system of equations

\[ \hat{p}_{b} = \frac{\pi_{b} \hat{M}_{b}}{\sum_{i=1}^{v} \pi_{i} \hat{M}_{i}}, \quad b = 1, 2, \ldots, v. \]  \hfill (67)

After simple transformations the above system takes the form

\[ (\hat{p}_{1} - 1) \pi_{1} \hat{M}_{1} + \hat{p}_{1} \pi_{2} \hat{M}_{2} + \cdots + \hat{p}_{v} \pi_{v} \hat{M}_{v} = 0 \]
\[ \hat{p}_{2} \pi_{1} \hat{M}_{1} + (\hat{p}_{2} - 1) \pi_{2} \hat{M}_{2} + \cdots + \hat{p}_{v} \pi_{v} \hat{M}_{v} = 0 \]
\[ \vdots \]
\[ \hat{p}_{v} \pi_{1} \hat{M}_{1} + \hat{p}_{v} \pi_{2} \hat{M}_{2} + \cdots + (\hat{p}_{v} - 1) \pi_{v} \hat{M}_{v} = 0, \]  \hfill (68)

where \( \hat{M}_{b} \) are unknown variables we want to find, \( \hat{p}_{s} \) are optimal transient probabilities determined by (59) and \( \pi_{s} \) are steady probabilities determined by (4).

Since the system of equations (68) is homogeneous and it can be proved that the determinant of its main matrix is equal to zero, then it has nonzero solutions and moreover, these solutions are ambiguous. Thus, if we fix some of the optimal values \( \hat{M}_{b} \) of the mean values \( M_{b} \) of the unconditional sojourn times at the operation states, for instance by arbitrary fixing one or a few of them, we may find the values of the remaining once and this way to get the solution of this equation.

Having this solution, it is also possible to look for the optimal values \( M_{b} \) of the mean values \( M_{b} \) of the conditional sojourn times at the operation states using the following system of equations

\[ \sum_{z=0}^{u} p_{b z} \hat{M}_{b} = \hat{M}_{b}, \quad b = 1, 2, \ldots, v, \]  \hfill (69)

obtained from (2) by replacing \( M_{b} \) by \( \hat{M}_{b} \) and \( M_{b} \) by \( M_{b} \), were \( p_{b} \) are known probabilities of the system operation process transitions between the operation states \( z_{a} \) in \( z_{a}, \ b = 1, 2, \ldots, v \).

Another very useful and much easier to be applied in practice tool that can help in planning the operation processes of the complex technical systems are the system operation process optimal mean values of the total system operation process sojourn times \( \hat{\theta}_{b} \) at the particular operation states \( z_{a} \), \( b = 1, 2, \ldots, v \), during the fixed system operation time \( z_{a} \) during the fixed system operation time \( z_{a} \), \( b = 1, 2, \ldots, v \).

The knowledge of the optimal values \( \hat{M}_{b} \) of the mean values of the unconditional sojourn times and the optimal values \( M_{b} \) of the mean values of the conditional sojourn times at the operation states and the optimal mean values \( \hat{M}_{b} \) of the total sojourn times at the particular operation states during the fixed system operation time may by the basis for changing the complex technical systems operation processes in order to ensure these systems operation more reliable.

**Example (continuation)**

We consider a series system \( S \) composed of the subsystems \( S_{1} \) and \( S_{2} \), with the scheme showed in
Figures 1-3. This system reliability structure and its components reliability parameters depend on its changing in time operation states with arbitrarily fixed the number of the system operation process states \( \nu = 4 \) and their influence on the system reliability indicated in Sections 2-3 where its main reliability characteristics are predicted.

To find the optimal values of those system reliability characteristics, we conclude that the objective function defined by (46), in this case, as the linear function (71) that according to (50)-(51) takes the form

\[
\mu(2) = p_1 \cdot 25.00 + p_2 \cdot 14.88 + p_3 \cdot 13.04 + p_4 \cdot 7.04.
\]  

(71)

Arbitrarily assumed, the lower \( \bar{p}_s \) and upper \( \hat{p}_s \) bounds of the unknown optimal values of transient probabilities \( p_s, b = 1,2,3,4 \), respectively are:

\[
\bar{p}_1 = 0.201, \quad \bar{p}_2 = 0.03, \\
\bar{p}_3 = 0.245, \quad \bar{p}_4 = 0.309, \\
\bar{p}_1 = 0.351, \quad \bar{p}_2 = 0.105, \\
\bar{p}_3 = 0.395, \quad \bar{p}_4 = 0.459.
\]  

(72)

Therefore, according to (47), we assume the following bound constraints

\[
0.201 \leq p_1 \leq 0.351, \quad 0.030 \leq p_2 \leq 0.105, \\
0.245 \leq p_3 \leq 0.395, \quad 0.309 \leq p_4 \leq 0.459. \\
\sum_{i=1}^4 p_i = 1.
\]  

(73)

Now, before we find optimal values \( \hat{p}_s \) of the transient probabilities \( p_s, b = 1,2,3,4 \), that maximize the objective function (71), we arrange the system conditional lifetime mean values \( \mu_s(2) \), \( b = 1,2,3,4 \), in non-increasing order

\[
\mu_1(2) \geq \mu_2(2) \geq \mu_3(2) \geq \mu_4(2).
\]

Further, according to (49), we substitute

\[
x_i = p_1, \quad x_i = p_2, \quad x_i = p_3, \quad x_i = p_4,
\]

and we maximize with respect to \( x_i, i = 1,2,3,4 \), the linear form (71) that according to (50)-(51) takes the form

\[
\mu(2) = x_1 \cdot 25.00 + x_2 \cdot 14.88 + x_3 \cdot 13.04 + x_4 \cdot 7.04,
\]

(77)

with the following bound constraints

\[
0.201 \leq x_i \leq 0.351, \quad 0.030 \leq x_i \leq 0.105, \\
0.245 \leq x_i \leq 0.395, \quad 0.309 \leq x_i \leq 0.459.
\]

(78)

According to (53), we calculate

\[
\bar{x} = \sum_{i=1}^4 x_i = 0.785, \\
\hat{x} = 1 - \bar{x} = 1 - 0.785 = 0.215
\]

(79)

and according to (54), we determine

\[
\bar{x} = 0, \quad \hat{x} = 0, \quad \bar{x} - \hat{x} = 0, \\
\bar{x} = 0.201, \quad \hat{x} = 0.351, \quad \bar{x} - \hat{x} = 0.150, \\
\bar{x} = 0.231, \quad \hat{x} = 0.456, \quad \bar{x} - \hat{x} = 0.225, \\
\bar{x} = 0.476, \quad \hat{x} = 0.851, \quad \bar{x} - \hat{x} = 0.375, \\
\bar{x} = 0.785, \quad \hat{x} = 1.31, \quad \bar{x} - \hat{x} = 0.525.
\]

(80)

From the above, as according to (79), the inequality (55) takes the form

\[
\bar{x} < 0.215,
\]

(81)

it follows that the largest value \( I \in \{0,1,2,3,4\} \) such that this inequality holds is \( I = 1 \).

Therefore, we fix the optimal solution that maximize linear function (77) according to the rule (57).

Namely, we get

\[
\bar{x} = 0.351, \quad \hat{x} = 0.030, \\
\bar{x} = 0.245, \quad \hat{x} = 0.309.
\]

(82)

Finally, after making the inverse to (74) substitution, we get the optimal transient probabilities

\[
\hat{p}_1 = \hat{x}_1 = 0.351, \quad \hat{p}_2 = \hat{x}_2 = 0.095, \\
\hat{p}_3 = \hat{x}_3 = 0.245, \quad \hat{p}_4 = \hat{x}_4 = 0.309.
\]

(83)
by the linear form (71) giving, according to (60) and (82), its optimal value

$$\hat{\mu}(2) = \hat{p}_1 \cdot 25.00 + \hat{p}_2 \cdot 14.88 + \hat{p}_3 \cdot 13.04 + \hat{p}_4 \cdot 7.04 = 0.351 \cdot 25.00 + 0.095 \cdot 14.88 + 0.245 \cdot 13.04 + 0.309 \cdot 7.07 \equiv 15.56. \quad (83)$$

Substituting the optimal solution (82) into the formula (61), we obtain the optimal solution for the mean values of the exemplary system unconditional lifetimes in the reliability state subsets \{1,2,3\} and \{3\}, that are as follows

$$\hat{\mu}(1) = \hat{p}_1 \cdot 27.78 + \hat{p}_2 \cdot 16.27 + \hat{p}_3 \cdot 14.82 + \hat{p}_4 \cdot 6.47 = 0.351 \cdot 27.78 + 0.095 \cdot 16.27 + 0.245 \cdot 14.82 + 0.309 \cdot 6.47 \equiv 17.31, \quad (84)$$

$$\hat{\mu}(3) = \hat{p}_1 \cdot 22.73 + \hat{p}_2 \cdot 13.71 + \hat{p}_3 \cdot 11.48 + \hat{p}_4 \cdot 6.47 = 0.351 \cdot 22.73 + 0.095 \cdot 13.71 + 0.245 \cdot 11.48 + 0.309 \cdot 6.47 \equiv 14.09, \quad (85)$$

and according to (64), the optimal values of the mean values of the system unconditional lifetimes in the particular reliability states 1, 2 and 3, respectively are

$$\hat{\mu}(1) - \hat{\mu}(2) = 1.75,$$

$$\hat{\mu}(2) - \hat{\mu}(3) = 1.47,$$

$$\hat{\mu}(3) = \hat{\mu}(3) = 14.09. \quad (86)$$

Moreover, according to (62)-(63), the corresponding optimal unconditional multistate reliability function of the system is of the form

$$\hat{R}(t, \cdot) = [1, \hat{R}(t, 1), \hat{R}(t, 2), \hat{R}(t, 3)] \quad (87)$$

for \( t \geq 0 \), with the coordinates given by

$$\hat{R}(t, 1) = 0.351 \cdot [R(t, 1)]^{1^{(3)}} + 0.095 \cdot [R(t, 1)]^{2^{(3)}} + 0.245 \cdot [R(t, 1)]^{3^{(3)}} + 0.309 \cdot [R(t, 1)]^{4^{(3)}},$$

$$\hat{R}(t, 2) = 0.351 \cdot [R(t, 2)]^{1^{(3)}} + 0.095 \cdot [R(t, 2)]^{2^{(3)}} + 0.245 \cdot [R(t, 2)]^{3^{(3)}} + 0.309 \cdot [R(t, 2)]^{4^{(3)}},$$

$$\hat{R}(t, 3) = 0.351 \cdot [R(t, 3)]^{1^{(3)}} + 0.095 \cdot [R(t, 3)]^{2^{(3)}} + 0.245 \cdot [R(t, 3)]^{3^{(3)}} + 0.309 \cdot [R(t, 3)]^{4^{(3)}}, \quad (88)$$

where \([R(t, 1)]^{b^{(3)}}, \ [R(t, 2)]^{b^{(3)}}, \ [R(t, 3)]^{b^{(3)}}, \ b = 1,2,3,4,\) are fixed in Section 3.

The graph of the exemplary system optimal reliability function \( R(t, \cdot) \) given by (87)-(88) is presented in Figure 6.

Figure 6. The graph of the exemplary system optimal reliability function \( R(t, \cdot) \) coordinates

As the critical reliability state is \( r = 2 \), then the exemplary system optimal system risk function, according to (65), is given by

$$\hat{r}(t) = 1 - \hat{R}(t, 2) \quad \text{for} \ t \geq 0, \quad (89)$$

where \( \hat{R}(t, 2) \) is given by (88).

Figure 7. The graph of the exemplary system optimal risk function \( \hat{r}(t) \)

Hence and considering (66), the moment when the optimal system risk function exceeds a permitted level, for instance \( \delta = 0.025 \), is

$$\tau = \hat{r}^{-1}(\delta) \equiv 2.55. \quad (90)$$

It can be seen that the optimal system reliability characteristics given by (87)-(88), (83)-(85), (86), (89) and (90) are better than that before optimization.
given respectively by (40)-(41), (42), (43), (44) and (45). Substituting the exemplary operation process optimal transient probabilities at operation states
\[
\hat{\rho}_i = 0.351, \quad \hat{\rho}_2 = 0.095, \quad \hat{\rho}_3 = 0.245, \quad \hat{\rho}_4 = 0.309,
\]
determined by (82) and the steady probabilities
\[
\pi_i \equiv 0.236, \quad \pi_2 \equiv 0.169, \quad \pi_3 \equiv 0.234, \quad \pi_4 \equiv 0.361,
\]
determined by (9) into (68), we get the following system of equations with the unknown optimal mean values \( \hat{M}_s \) of the exemplary system operation process unconditional sojourn times at the operation states we are looking for
\[
-0.153164\hat{M}_1 + 0.059319\hat{M}_2 + 0.082134\hat{M}_3 + 0.126711\hat{M}_4 = 0
\]
\[
0.02242\hat{M}_1 - 0.152945\hat{M}_2 + 0.02223\hat{M}_3 + 0.034295\hat{M}_4 = 0
\]
\[
0.057824\hat{M}_1 + 0.041405\hat{M}_2 - 0.17667\hat{M}_4 + 0.088445\hat{M}_4 = 0
\]
\[
0.072924\hat{M}_1 + 0.052221\hat{M}_2 + 0.072306\hat{M}_3 - 0.249451\hat{M}_4 = 0.
\] (91)

The determinant of the main matrix of the above homogeneous system of equations is equal to zero and therefore there are non-zero solutions of this system of equations that are ambiguous and dependent on one or more parameters. Thus, we may fix some of them and determine the remaining ones. To show the way of solving this system of equations, we may suppose that we are arbitrarily interested in fixing the value of \( \hat{M}_4 \) and we put \( \hat{M}_4 = 400 \). Further, substituting this value into the system of equations (91), we get
\[
-0.153164\hat{M}_1 + 0.059319\hat{M}_2 + 0.082134\hat{M}_3 = -50.6844
\]
\[
0.02242\hat{M}_1 - 0.152945\hat{M}_2 + 0.02223\hat{M}_3 = -13.7180
\]
\[
0.057824\hat{M}_1 + 0.041405\hat{M}_2 - 0.17667\hat{M}_4 = -35.3780
\]
\[
0.072924\hat{M}_1 + 0.052221\hat{M}_2 + 0.072306\hat{M}_3 = 99.7804
\]
and we solve it with respect to \( \hat{M}_1, \hat{M}_2 \) and \( \hat{M}_3 \), after omitting its last equation. This way obtained solutions that satisfy (91), are
\[
\hat{M}_1 \equiv 689, \quad \hat{M}_2 \equiv 261, \quad \hat{M}_3 \equiv 487, \quad \hat{M}_4 \equiv 400.
\] (92)

It can be seen that these solution differ much from the values \( M_1, M_2, M_3 \) and \( M_4 \) estimated in Section 2 by (8).

Having these solutions, it is also possible to look for the optimal values \( M_{\mu} \) of the mean values \( M_{\mu} \) of the exemplary system operation process conditional sojourn times at operation states. Namely, substituting the values \( \hat{M}_s \) instead of \( M_s \), the probabilities \( p_{s\mu} \) of the exemplary system operation process transitions between the operation states given in the matrix \([ p_{s\mu} ] \) defined by (6) and replacing \( M_{s\mu} \) by \( M_{\mu} \) in (69), we get the following system of equations
\[
0.22\hat{M}_{12} + 0.32\hat{M}_{13} + 0.46\hat{M}_{14} = 689
\]
\[
0.20\hat{M}_{21} + 0.30\hat{M}_{23} + 0.50\hat{M}_{24} = 261
\]
\[
0.12\hat{M}_{31} + 0.16\hat{M}_{33} + 0.72\hat{M}_{34} = 487
\]
\[
0.48\hat{M}_{41} + 0.22\hat{M}_{43} + 0.30\hat{M}_{44} = 400
\]

with the unknown optimal values \( M_{\mu} \) we want to find.

As the solutions of the above system of equations are ambiguous, then we fix some of them, say that because of practically important reasons, and we find the remaining ones. For instance:

- we fix in the first equation \( \hat{M}_{12} = 200, \quad \hat{M}_{13} = 500 \) and we find \( \hat{M}_{14} \equiv 1054 \);
- we fix in the second equation \( \hat{M}_{21} = 100, \quad \hat{M}_{23} = 100 \) and we find \( \hat{M}_{24} \equiv 422 \);
- we fix in the third equation \( \hat{M}_{31} = 900, \quad \hat{M}_{32} = 500 \) and we find \( \hat{M}_{34} \equiv 415 \);
- we fix in the fourth equation \( \hat{M}_{41} = 300, \quad \hat{M}_{42} = 500 \) and we find \( \hat{M}_{43} \equiv 487 \). (93)

It can be seen that these solutions differ greatly from the mean values of the exemplary system conditional sojourn times at the particular operation states before its operation process optimization given by (7).

Another very useful set of tools, which are much more easily applied in practice and which can help in planning the operation process of the system are the system operation process optimal mean values of the total sojourn times at the particular operation states during the system operation time that by the same assumption as in Section 2 is equal to \( \theta = 1 \text{ year} = 365 \) days.

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days. Under this assumption, after applying (70), we get the optimal values of the exemplary system operation process total sojourn times at the particular operation states during 1 year
\[
\hat{\bar{M}}_1 = \hat{E}[\hat{\theta}_1] = \hat{\bar{p}}, \hat{\theta} = 0.341 \cdot 365 \equiv 124.5, \\
\hat{\bar{M}}_2 = \hat{E}[\hat{\theta}_2] = \hat{\bar{p}}, \hat{\theta} = 0.105 \cdot 365 \equiv 38.3, \\
\hat{\bar{M}}_3 = \hat{E}[\hat{\theta}_3] = \hat{\bar{p}}, \hat{\theta} = 0.245 \cdot 365 \equiv 89.4, \\
\hat{\bar{M}}_4 = \hat{E}[\hat{\theta}_4] = \hat{\bar{p}}, \hat{\theta} = 0.309 \cdot 365 \equiv 112.8 \text{ days},
\]
that differ much from the values of \( \hat{\bar{M}}_1, \hat{\bar{M}}_2, \hat{\bar{M}}_3, \) \( \hat{\bar{M}}_4, \) determined by (11).

In practice, the knowledge of the optimal values of \( \hat{M}_1, \hat{M}_2, \hat{M}_3, \hat{M}_4, \) given respectively by (92), (93), (94), can be very important and helpful for the system operation process planning and improving in order to make the system operation more reliable. The comparison of the values of the exemplary system reliability characteristics before the system operation process optimization given by (42)-(43) and (45) with their values after the system operation process optimization respectively given by (83)-(86) and (90) justifies the sensibility of the performed system operation process optimization.

From the analysis of the results of the exemplary system operation process optimization it can be suggested to organize the system operation process in the way that causes the replacing (or the approaching/convergence to) the conditional mean sojourn times \( \bar{M}_0 \) of the system at the particular operation states before the optimization given by (7) by their optimal values \( \hat{M}_0 \) after the optimization given by (92).

An easier way might be to change the operation process characteristics that results in replacing (or the approaching/convergence to) the unconditional mean sojourn times \( M_m \) of the system at the particular operation states before the optimization given by (8) by their optimal values \( \hat{M}_m \) after the optimization given by (92).

Practically, the easiest way of the system operation process reorganizing might be to replace (or to approach/converge to) the total sojourn times \( \hat{\bar{M}}_0 \) of the system operation process at the particular operation states during the operation time \( \theta = 1 \text{ year} \) before the optimization given by (11) by their optimal values \( \hat{\bar{M}}_0 \) after the optimization given by (94).

The evaluation and optimization of the exemplary system operating at the varying operation conditions reliability are based on the arbitrary assumed input data. Therefore, the achieved results may only be considered as an illustration of the possibilities of applications of the proposed methods and procedures to this system operation and reliability analysis, prediction and optimization. However, the obtained evaluation may be a very useful example in real complex technical systems reliability optimization, especially during the design and when planning and improving the effectiveness of their operation processes.

6. Conclusion

The constructed general model of complex systems’ reliability, linking their reliability models and their operation processes models and considering variable at different operation states their reliability structures and their components reliability parameters was applied to the reliability evaluation of the exemplary system composed of a series-parallel and a series-“m out of l” subsystems linked in series. Next, the results of this model and the linear programming were applied to the optimization reliability and operation process of the considered exemplary system.

Presented in this paper tool is useful in reliability and operation optimization of a very wide class of real technical systems operating at the varying conditions that have an influence on changing their reliability structures and their components reliability parameters. The results can be interesting for reliability practitioners from various industrial sectors.

References


