Journal of Polish Safety and Reliability Association
Summer Safety and Reliability Seminars, Volume 5, Number 1, 2014

Franciszek Grabski
Naval University, Gdynia, Poland

Semi-Markov processes: application in system reliability and maintenance

Keywords

reliability, maintenance, semi-Markov process, semi-Markov perturbed processes

Abstract

The author's monograph "Semi-Markov Processes: Application in System Reliability and Maintenance" which will be published by Elsevier in 2014 is presented. The paper is composed of an introduction, the monograph contents, conclusions and the references the monograph contents is based on.

1. Introduction

The semi-Markov processes were introduced independently and almost simultaneously by P. Levy [61], W.L. Smith [79], and L. Takács [82] in 1954-55. The essential developments of semi-Markov processes theory were proposed by R. Pyke [71]-[73], E.Cinlar [14], Koroluk, Turbin [54]-[56], Limnios [63]. Here we only present semi-Markov processes with a discrete state space. A semi-Markov process is constructed by the Markov renewal process which is defined by the renewal kernel and the initial distribution or by another characteristics which are equivalent to the renewal kernel.

2. Monograph contents

The book entitled "Semi-Markov Processes: Applications in System Reliability and Maintenance" consists of the preface, 17 relatively short chapters, summary and bibliography.

In the introductory Chapter 1 the concept of a random (or stochastic) process is presented. The concept of a random process trajectory, *k*-th order distribution of the process, the process parameters like: the expected value, autocorrelation, covariance and correlation coefficient are discused. Moreover we introduce different kinds of stochastic processes such as a strict-sense stationary process, a wide-sense stationary (WSS) or a weak-stationary, the process with independent increments, a Markov process, an ergodic process. We also present some important processes like the Poisson process and the normal process. The relatively much space is devoted to the renewal process. The basic concepts and theorems of the renewal theory are presented in this chapter.

Chapter 2 is devoted to the discrete state space Markov processes especially continuous-time Markov processes and homogeneous Markov chains. The Markov processes are important class of the stochastic processes. The Markov property means that evolution of the Markov process in the future depends only on the present state and does not depend on the past history. The Markov process does not remember the past if the present state is given. Hence, the Markov process is called the process with *memoryless property*. This chapter covers some basic concepts, properties and theorems on homogeneous Markov chains and continuous-time homogeneous Markov processes with a discrete set of states. For continuous-time Markov processes the definition and basic properties of transition rate matrix are presented. From the properties of Markov process the Chapman-Kolmogorov deferential equations are derived. Basic definitions and properties of homogeneous Markov chain are also discussed there. For the reason of connection between the homogeneous Markov chain and the semi-Markov process, the classification of states and limiting distribution are discussed. The theory of that kind of processes allows to create models of real random processes, particularly in issues of reliability and maintenance.

Chapter 3 provides the definitions and basic properties related to a discrete state space semi-Markov process. The semi-Markov process is constructed by the so called Markov renewal process that is a special case the two-dimensional Markov sequence. The Markov renewal process is defined by the transition probabilities matrix, called the renewal kernel and an initial distribution or by another characteristics which are equivalent to the renewal kernel. The counting process corresponding to the semi-Markov process allows to determine concept of the process regularity. The process is said to be regular if the corresponding counting process has a finite number of jumps on a finite period. In the chapter are also shown the other methods of determining the semi-Markov process. The presented concepts are illustrated by some examples. Elements of the semi-Markov process statistical estimation are also presented in the chapter. Here there is considered estimation of the renewal kernel elements by observing one or many sample paths in the time interval, or given number of the state changes. Basic concepts of the non homogeneous semi-Markov processes theory are introduced.

Chapter 4 is devoted to some characteristics and parameters of the semi-Markov process. From the previous chapter it follows that a semi-Markov process is defined by renewal kernel and initial distribution of states or another equivalent parameters. Those quantities contain full information about the process and they allow us to find many characteristics and parameters of the process, that we can translate on the reliability characteristics in the semi-Markov reliability model. The cumulative distribution functions of the first passage time from the given states to subset of states and expected values and second moments corresponding to them are considered in this chapter. The equations for these quantities are here presented. Moreover, in the chapter there is discussed a concept of interval transition probabilities and the Feller equations are also derived. The interval transition probabilities for the alternating process for Poisson and Furry-Yule processes are derived.These equations allowed obtaining the interval transition probabilities for the alternating process and also for the Poisson and Furry-Yule processes. Karolyuk & Turbin theorems of the limiting probabilities are also presented in the chapter. Furthermore, the reliability and maintainability characteristics and parameters in semi-Markov models are considered in the chapter. At the end of the chapter there are shown numerical illustrative examples and proofs of some theorems.

The chapter 5 is concerned with the application of the perturbed semi-Markov processes in reliability problems. It is well known that in case of complex semi-Markov models usually the calculating of the exact probability distribution of the first passage time to the subset of states, is very difficult. Then, it seems that the only way is to find the approximate probability distribution of that random variable. It is possible by using the results coming from the theory of semi-Markov processes perturbations. The perturbed semi-Markov processes are defined in different way by different authors. This theory has a rich literature. The most significant and original results include the books of Korolyuk and Turbin [54]-[56], Korolyuk and Limnios [59], Gyllenberg and Silvestrov [38]. We can find many interesting results concerning of perturbed semi-Markov processes in papers of Gercbakh [22]-[23], Pavlov & Ushakov [70], Shpak [75], Gyllenberg & Silvestrov [37], Domsta & Grabski [17] and many more. In this Chapter we present only a few the simplest types of perturbed SM processes. First of them is presented by Shpak, the second one is introduced by Pavlov and Ushakov and was presented by Gercbakh. The third one is defined by Koryoluk and Turbin. All concepts of the perturbed SM processes are explained in the same simple example. Moreover, an exemplary approximation of the system reliability function with illustrative numerical example is presented in this chapter. The last section is devoted to the state space aggregation method.

In chapter 6 the random processes determined by the characteristics of the semi-Markov Process are considered. First of them is a renewal process generated by return times of a given state. The systems of equations for the distribution and expectation of them have been derived. The limit theorem for the process is formulated by adoption a theorem of the renewal theory. The limiting properties of the alternating process and integral functionals of the semi-Markov process are also presented in this chapter. The chapter contains illustrative examples.

The Semi-Markov reliability model of two different units renewable cold standby system and the SM model of a hospital electrical power system consisting of mains, an emergency power system and the automatic transfer switch with the generator starter is discussed in chapter 7. The renewable cold standby system with series *N* components exponential subsystems is also presented here. The embedded Semi-Markov process concept is applied for description of the system evolution. In our models time to failure of the system is represented by a random variable denoting the first passage time from the given state to the subset of states. The appropriate theorems of the Semi-Markov processes theory allow us to evaluate the reliability function and some reliability characteristics. In case of difficulties of calculation an exact reliability function of the system by means of the Laplace transform, we propose application the theorem of Semi-Markov processes perturbation theory that enable us to get an approximate reliability function of the system. Some illustrative examples in this chapter allow to explain presented concepts.

In chapter 8 the model of the multi-stage operation is presented. The operation processes like transport processes, production processes and many other consist of some stages which are realised in turn. In each stage there are some possible failures that lead to perturbation or to failure of the operation. In the chapter the model of multi-stage operation without repair and the model with repair are constructed. Application of semi-Markov process theory results allowed to calculate the reliability parameters and characteristics of the multistage operation. The models are applied for modeling the multi stage transport operation processes.

In chapter 9 the semi-Markov model of the load rate process is discussed. Speed of car, load rate of ship engine are example of the random load rate process. The construction of discrete state model of the random load rate process with continuous trajectories leads to the semi-Markov random walk. Estimation of the model parameters and calculating the semi-Markov process characteristics and parameter give us possibility to analyse the semi-Markov load rate. Chapter 10 contains the semi-Markov model of the multi-task operation process. The outline of the chapter is: description and assumptions, construction of the semi-Markov model, calculating the reliability characteristics, calculating the approximate reliability function, an example. Chapter 11 is devoted to the semi-Markov failure rate process. In this chapter the failure rate is assumed to be a stochastic process with non-negative and right continuous trajectories. The reliability function is defined as an expectation of a function of that random process. Particularly the failure rate is defined by the discrete state space semi-Markov process. The theorem concerning with the renewal equations for the conditional reliability function with a semi-Markov process as a failure rate is proved. Reliability function with a random walk as a failure rate is investigated. For Poisson failure rate process and Furry-Yule failure rate process the reliability functions are obtained. In chapter 12 there is presented the semi-Markov model of the renewal series system coming from Cinlar paper. A lot of system reliability characteristics are calculated. The chapter one can treat as a teaching part.

In chapter 13 time to a preventive service optimization problem is formulated. The semi-Markov model of the operation process allowed to formulate the optimization problem. A theorem containing the sufficient conditions of the existing solution is formulated and proved. An example explains and illustrates the presented problem.

In chapter 14 a semi-Markov model of a system component damage is discussed. Presented here models deal with unrepairable systems. The multi-state reliability functions and corresponding expectations, second moments and standard deviations are evaluated for the presented cases of the component damage. A special case of the model is a multi-state model with two kind of failures. A theorem dealing with the inverse problem for simple damage exponential model is formulated and proved.

In Chapter 15 some results of investigation of the multistate monotone system with components modelled by the independent semi-Markov processes are presented. Many papers are devoted to the reliability of multistate monotone systems [4], [46]-[48], [56]. Many other concern the semi-Markov models of multistate systems [33], [53], [64]-[65]. The chapter is organized as follows. The second section contains a basic notation, concepts and assumptions. Particularly it deals with a system structure and a concept of a multistate monotone system. The third section is devoted to the unrepairable system components. We assume that the states of system components are modelled by the independent semi-Markov processes. Some characteristics of a semi-Markov process are used as reliability characteristics of the system components. In the next section the binary representation of the multistate monotone systems is discussed. The concept of a minimal path vector to level l is crucial to these considerations. The multistate reliability functions of the system components and the whole system are discussed in the section. The last section provides the semi-Markov model of the renewable multistate system. The probability distribution and the limiting distribution of the systems reliability state are computed. The presented concepts and models are illustrated by some numerical examples.

Semi-Markov maintenance nets were introduced by Silvestrov [76]-[78]. In [31] were presented semi-Markov models of the simple maintenance nets. The semi-Markov models of functioning the maintenance systems which are called maintenance nets are presented in Chapter 16. Elementary maintenance operations form the states of SM model. Some concepts and results of Semi-Markov process theory provide the possibility of computing important characteristics and parameters of maintenance process. There are discussed two semi-Markov models of maintenance nets in the chapter.

In chapter 17 there are presented basic concepts and results of the theory of semi-Markov decision processes. The algorithm of optimization a SM decision process with a finite number of state changes is discussed here. The algorithm is based on a dynamic programming method. To clarify it the

SM decision model for the maintenance operation is shown. The optimization problem for the infinite duration SM process and the Howard algorithm which enables to find the optimal stationary strategy is also discussed here. To explain this algorithm a decision problem for renewable series system is presented.

3. Conclusions

The stochastic processes theory allows to construct models of the real random processes which are investigated in different fields of science.

This book is a modern view of discrete state space and continuous time semi-Markov processes and their applications in reliability and maintenance. A semi-Markov stochastic process has the memoryless property at the moments of the state changes. It means that a state of the process and its sojourn time depends only on a previous state but does not depend on the past states and their sojourn times. The semi-Markov process keeps on half-intervals constant values coming from a discrete state space and its trajectories are the right-continuous real functions.

Usually the semi-Markov process is defined by a Markov renewal process which is a special case of the two dimensional Markov chains. The transition probabilities of the Markov renewal process form the matrix which is called a renewal kernel. The semi-Markov process is defined if the renewal kernel is given. This process can be defined in a different way, by another characteristics that are equivalent to the renewal kernel.

The semi-Markov processes theory allows us to construct a lot of models of evolution the reliability and maintenance systems over time. We should remember that a semi-Markov process may be a stochastic model of a real process only if it satisfies the conditions of definition of the semi-Markov process.

In this book we try to answer the following questions: How can we construct the semi-Markov models of the systems reliability and maintenance? What kind of reliability parameters and characteristics can we obtain from those models? To answer those questions we will present semi-Markov models from field of reliability and maintenance. The presented models give an answer to these and other questions.

The book is a very helpful resource for understanding the basic concepts and results of semi-Markov processes theory.

The book is primarily intended to researchers and scientists dealing with mathematical reliability theory (mathematicians) and practitioners (engineers) dealing with the reliability analysis and tool for the scientists, the PhD students and MSc students of the technical universities and research centers.

References

[1] Anisimov, B.B. (1970). Multidimensional limit theorems for semi-Markov processes. *Probability Theory and Mathematical Statistics.* 3, 3-21 (in Rusian).

[2] Anisimov, B.B. (1970). Limit theorems for semi-Markov processes. *Probability Theory and Mathematical Statistics* 2, 3-15 (in Rusian).

[3] Asmussen, S. (1987). *Applied Probability and Queues.* Chichester: Wiley.

[4] Aven, T. (1985). Relibility evaluation of multistate systems with multistate components. *IEEE Transactions on Reliability* 34(2), 463-472.

[5] Baker, M.C.J. (1990). How often should a machine be inspected. *International Journal of Quality Reliability Management* 7, 14-18.

[6] Barlow, R.E. & Proshan, F. (1965). *Mathematical theory of reliability.* New York, London, Sydney; Wiley.

[7] Barlow, R.E. & Proshan, F. (1975). *Statistical theory of reliability and life testing.* New York: Holt, Rinchart and Winston.

[8] Billingsley, P. (1979). *Probability and Measure.* John Wiley & Sons.

[9] Block, H.W. & Savits, T.H. (1982). A decomposition for multistate monotone systems. *Jouranal of Applied Probability* 19; 2, 391-402.

[10] Bobrowski, D. (1985). *Mathematical Models in Reliability Theory* (in Polish). WNT, Warsaw.

[11] Brodi, S.M & Vlasenko, O.N. (1969). Reliability of systems with many kind of work. Reliability theory and queue theory. *Nauka* 165-171 (in Russian).

[12] Brodi, S.M. & Pogosian, J.A. (1973). Embedded stochastic processes in theory of queue. *Naukova Dumka* (in Russian).

[13] Ciesielski, Z. (1991). Asymptotic Nonparamertic Spline Density Estimation. *Probab. Math. Statist.* Vol. 12, 1-24.

[14] Cinlar, E. (1969). Markov renewal theory. *Adv. Appl. Probab.* No 2, 1, 123-187.

[15] Csenki, A. (1994). *Dependability for Systems with a Partitioned State Space Markov and Semi-Markov Theory and Computatinal Implemantation.* New York; Springer-Verlang Inc.

[16] Doob, J.L. (1953). *Stochastic Processes.* New York: John Wiley & Sons, London: Chapman & Hall.

[17] Domsta, J. & Grabski, F. (1995). The first exit of almost strongly recurrent semi-Markov processes. *Applicationes Mathematicae* 23; No. 3, 285-304.

[18] Epanechnikov, V.A. (1969). Non-parametric estimation of multivariate probability density. *Theory of Probability and its Application* 85(409), 66-72.

[19] Feller, W. (1964). On semi-Markov processes. *Proc. Nat. Acad. Sci.* 51; No. 4, 653-659.

[20] Feller, W. (1968). *An Introduction to probability Theory and Its Applications*. Vol. 1. Third Edition. New York, London, Sydney: John Wiley & Sons.Inc.

[21] Feller, W. (1966). *An Introduction to Probability Theory and Its Applications*. Vol.2. New York, London, Sydney: John Wiley & Sons.Inc.

[22] Gertsbakh. I.B. (1969). Models of preventive service. *Sovetskoe radio*. (in Russian).

[23] Gertsbakh, I.B. (1984). Asymptotic methods in reliability theory: a review. *Adv. Appl. Prob.* 16, 147-175.

[24] Gichman, I.I. & Skorochod, A.W. (1968). *Introduction to the Theory of Stochastic processes.* Warsaw; PWN (in Polish).

[25] Girtler, J. & Grabski, F. (1989). Choice of optimal time to preventive service of the ship combustion engine. *Scientific Problems of Machines Operation and Maintenance* 4, 481-489 (in Polish).

[26] Gniedenko, B.W., Bielajew, J.K. & Sołowiew, A,D. (1968). *Mathematical Method in Reliability Theory*. Warsaw: WN-T (in Polish).

[27] Grabski, F. (1981). Analize of random working rate based on semi-Markov processes. *Scientifi Problems of Machines Operation and Maintenance* 3-4, (47-48), 294-305 (in Polish).

[28] Grabski, F. (1982). *Theory of Semi-Markov Operation Processes*. ZN AMW; 75A; Gdynia (in Polish).

[29] Grabski, F. Piaseczny, L. & Merkisz, J. (1999). Stochastic models of the research tests of the toxicity of the exhaust gases. *Journal of KONES International Combustion Engines* Vol.6 No. 1-2, 34-41 (in Polish)

[30] Grabski, F. (2002). *Semi-Markov models of reliability and operation.* Warsaw : IBS PAN (in Polish).

[31] Grabski, F. (2003). *Some problems of modeling transport systems.* Warsaw-Radom; ITE (in Polish).

[32] Grabski, F. (2007). Application of semi-Markov processes in reliability. *Electronic Journal, Reliability: Theory & Applications* Vol.2, No.3-4, 60-75.

[33] Grabski, F. & Kołowrocki, K. (1999). Asymptotic Reliability of Multistate System with semi-Markov states of components. *Proc. of the European Conference on Safety and Reliability - ESREL'99. Safety and Reliability,* 317-322.

[34] Grabski, F. (2003). The reliability of an object with semi-Markov failure rate. *Applied Mathematics and Computation* 135, 1-16.

[35] Grabski, F. (2011). Semi-Markov failure rate process. *Applied Mathematics and Computation* 217, 9956-9965.

[36] Grabski, F. & Jaźwiński, J. (2009). *Functions of Random Variables in Problems of Reliability, Safety and Logistics.* Warsaw; WKŁ; 2009 (in Polish).

[37] Gyllenberg, M. & Silvestrov, D.S. (1999). Quasi-stationary phenomena for semi-Markov processes. In: Janssen J, Limnios N. (eds) *Semi-Markov Models and Applications* 33-60.

[38] Gyllenberg, M. & Silvestrov, D.S. (2008). Quasi-Stationary Phenomena in Nonlinearly Perturbed Stochastic System. *De Gruyter Expositions in Mathematics,* 44. XII+579.

[39] Harriga, MA. (1996). A maintenance inspection model for a single machine with general failure distribution functions. *Pacific Journal of Mathematics* 31(2), 403-415.

[40] Howard, R.A. (1960). *Dynamic Programing and Markov Processes.* Cambridge, Massachusetts; MIT Press.

[41] Howard, R.A.. (1964). Research of semi-Markovian decision structures. *Journal of Operations Research Society* 6, 163-199.

[42] Howard, R.A. (1971). *Dynamic probabilistic system. Vol. II: Semi-Markow and decision processes.* New York, London, Sydney, Toronto; Wiley.

[43] Iosifescu, M. (1980). *Finite Markov Processes and Their Applications.* John Wiley & Sons, Ltd.

[44] Jewell, W.S. (1963). Markov-renewal programming. *Operation Research* 11, 938-971.

[45] Kato, T. (1966). *Perturbation Theory for Linear Operators.* Berlin; Springer.

[46] Kołowrocki, K. (2001). On limit reliability functions of large multi-state systems with ageing components. *Applied Mathematics and Computation* 121, 313-361.

[47] Kołowrocki, K. (2004). *Reliability of Large Systems*. Elsevier.

[48] Kołowrocki, K. & Soszyńska-Budny, J. (2011). Asymptotic approach to reliability of large complex systems. *Electronic Journal Reliability: Theory & Applications* 2(4), 103-128.

[49] Kołowrocki, K. & Kuligowska, E. (2013). Monte Carlo simulation application to reliability evaluation of port grain transportation system operating at variable conditions. *Journal of Polish Safety and Reliability Association, Summer Safety and Reliability Seminars* 4(1), 73-82.

[50] Kopociński, B. (1973). *Outline of the renewal theory and reliability.* Warsaw: PWN (in Polish).

[51] Kopocińska, I. & Kopociński, B. (1980). On system reliability under random load of elements. *Aplicationes Mathematicae* XVI,1, 5-14.

[52] Kopocińska, I. (1984). The reliability of an element with alternating failure rate. *Aplicationes Mathematicae* XVIII, 2, 187-194.

[53] Korczak, E. (1997). Reliability analisis of non-repaired multistste systems. *Advances in Safety and Reliability: ESREL'9*,2213-2220.

[54] Korolyuk, V.S. & Turbin, A.F. (1976). Semi-Markov processes and their applications. *Naukova Dumka* (in Russian).

[55] Korolyuk, V.S. & Turbin, A.F. (1982). Markov Renewal Processes in Problems of Systems Reliability. *Naukova Dumka* (in Russian).

[56] Korolyuk, V.S. & Turbin, A.F. (1993). *Decompositions of Large Scale Systems*. Kluwer Academic, Singapore.

[57] Korolyuk, V.S. & Swishchuk, A. (1995). *Semi-Markov Random Evolutions*. Derdrecht, Boston, Lomdon; Kluwer Academic .

[58] Korolyuk, V.S., Polishchuk, L.I. & Tomusyak, A.A. (1969). A limit theorem for semi-Markov processes. *Cybernetic*. Vol. 5, Issue 4, 524-526.

[59] Korolyuk, V.S. & Limnios, N. (2005). *Stochastic System in Merging Phase Space*. Singapore; World Scientific.

[60] Knopik, L. (2007). Some remarks on mean time between failures of repairable systems. *Proc. Summer Safety and Reliability Seminars* Vol. 2, 207-211.

[61] Lev'y, P. (1954). Proceesus semi-markoviens. Proc. Int. Cong. *Math. Amsterdam* 416-426.

[62] Letkowski, J. (2013). Application of the Poisson Process Distribution www.aabri.com/SA12Manuscripts/SA12083

[63] Limnios, N. & Oprisan, G. (2001). *Semi-Markov Processes and Reliability*. Boston; Birkhauser.

[64] Lisnianski, A. & Frankel, I. (2009). Non-Homogeneous Markov Reward Model for an Ageing Multi-State System under Minimal Repair. *International Journal of Performability Engineering* 4, 5, 303-312.

[65] Lisnianski, A. & Levitin, G. (2003). *Multi-State System Reliliability: Assesment, Optimization and Applications*. London, Sigapore: World Scientic.

[66] Moore, E.H. & Pake, R. (1968). Estimation of the transition distributions of a Markov renewal process. *Ann. Inst. Statist. Math*. 20, 411-424.

[67] Mine, H. & Osaki, S. (1970). *Markovian Decision Processes*. New York; AEPCI.

[68] Papadopoulou, A. & Vassiliou, P.C.G. (1999). Continuous time non-homogeneous semi-Markov systems. Semi-Markov models and applications. *Kluver Acad. Publ*., 241-251.

[69] Papoulis, A. (1991). *Probability, Random Variables, and Stochastic Processes*. New York: McGraw-Hill, Inc.

[70] Pavlov, I.V. & Ushakov, I.A. (1978). The asymptotic distribution of the time until a semi-Markov process gets out of the kernel. *Engineering Cybernetics* 2(3).

[71] Pyke, R. (1961). Markov renewal processes: definitions and preliminary properties. *Ann. of Math. Statist.*32, 1231-1242.

[72] Pyke, R. (1961). Markov renewal processes with finitely many states. *Ann. of Math. Statist*. 32, 1243-1259.

[73] Pyke, R. & Schaufele, R. (1964). Limit theorems for Markov renewal processes. *Ann. of Math. Statist*. 32, 1746-1764.

[74] Shiryayev, A.N. (1984). *Probability*. New York, Berlin, Heidelberg, Tokyo; Springer-Verlag.

[75] Shpak, W.D. (1971). On some approximation for calculation of a complex system reliability. *Cybernetics* 10, 68-73 (in Russian).

[76] Silvestrov, D.C. (1980). Semi-Markov processes with a discrete state space (in Russian). *Sovetskoe Radio*.

[77] Silvestrov, D.C. (1980). Semi-Markov processes with a discrete state space (in Russian). *Sovetskoe Radio*.

[78] Silvestrov, D.C. (2008). Nonlinearly Perturbed Stochastic Processes. *Theory of Stochastic Processes*. Vol. 14 (30), No.3-4, 129-164.

[79] Smith, W.L. (1955). Regenerative stochastic processes. London: *Proc. Roy. Soc. Ser*. A, 232,6-31.

[80] Solovyev, A.D. (1979). *Analytical Methods of Reliability Theory*. Warsaw: WN-T.

[81] Taga, Y. (1963). On the limiting distributions in Markov renewal processes with finitely many states. *Ann. Inst. Statist. Math*. 15, 1-10.

[82] Takács, L. (1954). Some investigations concerning recurrent stochastic processes of a certain type. *Magyar Tud. Akad. Mat. Kutato Int. Kzl*. 3, 115-128.

[83] Takács, L. (1959). On a sojourn time problem in the theory of stochastic processes. *Trans. Amer. Math. Soc*. 93, 531-540.

[84] Zając, M. (2007). *Reliability model of the inter-model system*. PhD thesis. Wroclaw University of Technology.