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Application of perturbed semi-Markov processes in reliability

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semi-markov perturbed process, approximate reliability function, asymptotic

Abstract

The paper is concerned with the application of perturbed semi-Markov (SM) processes in reliability problems. There are two kinds of perturbed SM processes presented in the paper. First of them was defined by Shpak and the second one was introduced by Pavlov and Ushakov. Shpak's concept of perturbed SM is applied for calculating the approximate reliability function of many tasks operation process and Pavlov and Ushakov concept of that one is used to obtain the approximate reliability function of a repairable cold standby system with a switch.

1. Introduction

It is well known that in case of complex semi- Markov models the calculating of the exact probability distribution of the first passage time to subset of states, usually is very difficult. Then, the only way it seems to be is finding the approximate probability distribution of that random variable. It is possible by using the results from the theory of semi-Markov processes perturbations. The perturbed semi-Markov processes are defined in different ways by different authors [8], [9], [10], [11], [4]. We introduce only two concepts of the perturbed semi-Markov process presented – the concept of Shpak and the concept of Pavlov & Ushakov which was presented by [4].

2. Characteristics of Semi-Markov process

To have a semi-Markov process as a model we have to define its initial distribution and all elements of its kernel. Recall that the semi-Markov kernel is the matrix of transition probabilities of the Markov renewal process.

$$\mathrm{Q}(t) = [Q_{ij}(t) : i, j \in S], t \geq 0 \tag{1}$$

where

$$Q_{ij}(t) = P(\tau_{n+1} - \tau_n \leq t, X(\tau_{n+1}) = j \mid X(\tau_n) = i) \tag{2}$$

and $\{\tau_n : n = 0, 1, ...\}$ denotes a sequence of state changes instants. From the definition of semi-Markov process it follows that the sequence $\{X(\tau_n) : n = 0, 1, ...\}$ is a homogeneous Markov is with transition probabilities

$$p_{ij} = P(X(\tau_{n+1}) = j \mid X(\tau_n) = i) = \lim_{t \to \infty} Q_{ij}(t). \tag{4}$$

The function

$$\begin{aligned} G_i(t) &= P(\tau_{n+1} - \tau_n \leq t \mid X(\tau_n) = i) \\ &= \sum_{j \in S} Q_{ij}(t) \end{aligned} \tag{5}$$

is a cumulative probability distribution of a random variable T_i that is called a waiting time of the state i. The waiting time T_i is the time spent in state i when the successor state is unknown. The function

$$\begin{aligned} F_{ij}(t) &= P(\tau_{n+1} - \tau_n \leq t \mid X(\tau_n) = i, X(\tau_{n+1}) = j \\ &= \frac{Q_{ij}(t)}{p_{ij}} \end{aligned} \tag{6}$$

is a cumulative probability distribution of a random variable T_{ij} that is called a holding time of a state i, if the next state will be j. From (6) we have

$$Q_{ij}(t) = p_{ij} F_{ij}(t). \qquad (7)$$

It means that a semi- Markov process with a discrete state space can be defined by the transition matrix of the embedded Markov chain:

$$P = [p_{ij} : i, j \in S]$$

and a matrix of CDF of holding times:

$$F(t) = [F_{ij}(t) : i, \ j \in S].$$

A value of a random variable

$$\Delta_A = \min\{n \in \mathrm{N} : X(\tau_n) \in A\} \qquad (8)$$

denotes a discrete time (a number of state changes) of *a first arrival* at the set of states

$A \subset S$ of the embedded Markov chain $\{X(\tau_n) : n \in \mathrm{N}_0\}$. A random variable $\Theta_A = \tau\Delta_A$ denotes a first passage time to the subset *A* or the time of *a first arrival* at the set of states $A \subset S$ of the semi-Markov process $\{X(t) : t \geq 0\}$.
A function

$$\Phi_{iA}(t) = P(\Theta_A \leq t \mid X(0) = i), t \geq 0$$

is Cumulative Distribution Function (CDF) of a random variable Θ_{iA} denoting the first passage time from the state $i \in A'$ to a subset *A* or the exit time of $\{X(t) : t \geq 0\}$ from the subset A' with the initial state *i*. From the theorems presented by Koroluk & Turbin (1976), Silvestrov (1980), Grabski (2002) concerning the distributions and parameters of the random variables Θ_{iA} it follows that the Laplace- Stieltjes (L- S) transforms of the CDF $\Phi_{iA}(t)$, $i \in A'$ satisfy the linear system of equations

$$\tilde{\phi}_{iA}(s) = \sum_{j \in A} \tilde{q}_{ij}(s) + \sum_{k \in A'} \tilde{q}_{ik}(s) \tilde{\phi}_{kA}(s), \qquad (9)$$

where

$$\tilde{\phi}_{iA}(s) = \int_0^\infty e^{-st} d\Phi_{iA}(t), \qquad (10)$$

are L-S transforms of the unknown CDF of the random variables $\Theta_{iA}, i \in A'$ and

$$\tilde{q}_{iA}(s) = \int_0^\infty e^{-st} dQ_{ij}(t), \qquad (11)$$

That linear system allows to construct the system of linear equations for the expectation of the random variables $\Theta_{iA}, i \in A'$. The system is equivalent to the matrix equation

$$(\boldsymbol{I} - \boldsymbol{P}_{A'}) \overline{\Theta}_{A'} = \overline{\boldsymbol{T}}_{A'}, \qquad (12)$$

where

$$\boldsymbol{P}_{A'} = [p_{ij} : i, \ j \in A'],$$

$$\overline{\Theta}_{A'} = [E(\Theta_{iA}) : i, \in A']^T,$$

$$\overline{\boldsymbol{T}}_{A'} = [E(T_i) : i, \in A'],$$

and $\boldsymbol{I}$ is the unit matrix.

3. SM Perturbed processes by Shpak

In the paper of Shpak a perturbed SM process is called an associated with SM process. We introduce our version of Shpak definition. Let $A' = \{1, 2, ..., N\}$, $A = \{0\}$ and $S = A \cup A'$.
Suppose that $\{X^0(t) : t \geq 0\}$ be a semi-Markov process with the state space *A* and a kernel

$$\boldsymbol{Q}^0(t) = [Q_{ij}^0(t) : i, \ j \in A'].$$

Definition 1.
The semi-Markov process $\{X(t) : t \geq 0\}$ with a state space *S* is said to be perturbed with respect to the process $\{X^0(t) : t \geq 0\}$, if the components of the kernel $\boldsymbol{Q}(t) = [Q_{ij}(t) : i, j \in S]$ of the process $\{X(t) : t \geq 0\}$ are

$$Q_{ij}(t) = \begin{cases} \int_0^t [1 - F_i(x)] dQ_{ij}^0(x), & i, \ j \in A' \\ 0, & i \in A, \ j \in A' \end{cases}$$

$$Q_{i0}(t) = \int_0^t [1 - G_i^0(x)] dF_i(x), i \in A',$$

where $G_i^0(t) = \sum_{j \in A'} Q_{ij}^0(t)$ and the functions $F_i(t) = P\{Z_i \leq t\}, i \in A'$ are CDF of the independent random variables $Z_i, i \in A'$ having finite first moments.

A random variable Θ_{iA} denotes a first passage times from a state $i \in A'$ to state 0. Let

$$m_i^0 = \int_0^\infty [1 - G_i^0(t)]dt, \, i \in A', \tag{13}$$

where

$$G_i^0(t) = \sum_{j \in A'} Q_{ij}^0(t), \tag{14}$$

A number m_i^0 is an expectation of holding time of a state $i \in A'$ for the process $\{X^0(t) : t \geq 0\}$. A number $\varepsilon_i = p_{i0} = \lim_{t \to \infty} Q_{i0}(t)$ denotes a transition probability of the SM process $\{X(t) : t \geq 0\}$ from the state $i \in A'$ to the state 0. Let $\pi^0 = [\pi_1^0, ..., \pi_N^0]$ denotes a stationary distribution of an embedded Markov chain of the SM process SM $\{X^0(t) : t \geq 0\}$ and let

$$\varepsilon = \sum_{i \in A'} \pi_i^0 \varepsilon_i. \tag{15}$$

Theorem 1. [12]
If the embedded Markov chain of the SM process SM $\{X^0(t) : t \geq 0\}$ has the stationary distribution $\pi^0 = [\pi_1^0, ..., \pi_N^0]$, $\varepsilon > 0$ and distribution of waiting times, defined by CDF's $G_i^0(t)$, $i = 1, ..., N$, have positive finite expectations, then

$$\lim_{\varepsilon \to 0} P\{\varepsilon \Theta_{iA} > t\} = \exp[-\lambda \, t], \, t \geq 0, \tag{16}$$

where

$$\lambda = \frac{1}{\sum_{i \in A'} \pi_i^0 m_i^0}. \tag{17}$$

Notice, the asymptotic distribution of a random variable does not depend on the state $i \in A'$. If A' denotes a set of „up" states in SM reliability model $\{X(t) : t \geq 0\}$, and $A = \{0\}$ is a "down" state then this theorem allows to calculate the approximate reliability function. From above theorem it follows that

$$R(t) = P\{\Theta_{iA} > t\} = P\{\varepsilon \Theta_{iA} > \varepsilon t\} \approx \exp[-\lambda \varepsilon \, t], \, t \geq 0, \tag{18}$$

if ε is small number. We apply this theorem to calculate the approximate reliability function of the object making many tasks operation.

3.S M model of many tasks operation process

3.1. Description and assumptions

Many technical objects are destined to realization of many tasks, for example transport means. Different tasks determine different load rates and finally imply different failure rates. We assume that an object can realized tasks $z_1, ..., z_r$ that are the values of a random variable Z having a discrete distribution $P(Z = z_k) = a_k$, $k = 1, ..., r$. We suppose that at the moments τ_{2n}, $n \in \mathrm{N}_0$ with the probability a_k, $k = 1, ..., r$ are beginnings of the realizations of tasks z_k, $k = 1, ..., r$. Time of the task z_k realization is a nonnegative random variable ξ_k with CDF $U_k(x) = P(\xi_k \leq x)$. A failure of a working object may occur. Suppose that a lifetime of the object realizing task z_k is a nonnegative random variable ς_k with CDF $F_k(x) = P\{\varsigma_k \leq x\}$. In case of the failure of the object is repaired by time γ that is a positive random variable with CDF $G(x) = P(\gamma \leq x)$. If in time interval of length ξ_k the failure of the object does not occur then at the moment of the end of the task realization, a period of the object maintenance service begins. The length of that period is a nonnegative random variable η_k having distribution given by CDF $V_k(x) = P(\eta_k \leq x)$. Suppose that every service is full renewal and the instants of their ending are the moments of realization tasks beginning. We assume that presented random variables are mutually independent and their copies are independent too. We also suppose that at least the random variables ς_k, $k = 1, ..., r$ have absolutely continuous (with respect to the Lebesque measure) probability distribution and all random variables have finite and positive second moments.

3.2. Construction of a model

We define following states:

- k - a realization of the task z_k, $k = 1, ..., r$,
- $r + k$ – maintenance service after realization of the task z_k, $k = 1, ..., r$,
- 0 - general repair after failure.

Let $\{X(t): t \geq 0\}$ be a stochastic process with a piecewise constant, right continuous trajectories and a set of states $S = \{0, 1, ..., 2r\}$.

The change of process states takes place at the random instants $\tau_0, \tau_1, \tau_2, \ldots$.

From the assumptions it follows that a state of process that is achieved at an instant τ_n and its sojourn time does not depend on the states achieved at the instants $\tau_0, ..., \tau_{n-1}$ an their sojourn times. It means that $\{X(t): t \geq 0\}$ is a semi-Markov process. We assume that the initial distribution is given by

$$P(X(0)=k) = \begin{cases} a_k, & k = 1, ..., r \\ 0, & k = 0, r+1, ..., 2r. \end{cases}$$

From the definition of the semi-Markov kernel elements and assumptions we have

$$Q_{k\,r+k}(t) = P(\xi_k \leq t, \varsigma_k > \xi_k)$$

$$= \int_0^t [1 - F_k(t)] dU_k(x), \; k = 1, ..., r$$

$$Q_{k\,0}(t) = P(\varsigma_k \leq t, \xi_k > \varsigma_k)$$

$$= \int_0^t [1 - U_k(t)] dF_k(x), \; k = 1, ..., r,$$

$$Q_{r+k\;j}(t) = P(\eta_k \leq t, Z = z_j)$$

$$= a_j V_k(t), \; k = 1, ..., r, \; j = 1, ..., r,$$

$$Q_{0\,j}(t) = P(\gamma_k \leq t, Z = z_j)$$

$$= a_j G(t), \quad j = 1, ..., r.$$

Therefore the semi-Markov model is constructed. For simplicity of notation we take $r = 2$. In this case the Semi-Markov kernel of the model is

$$\mathrm{Q}(t) = \begin{bmatrix} Q_{01}(t) & Q_{02}(t) & 0 & 0 & 0 \\ Q_{10}(t) & 0 & 0 & Q_{13}(t) & 0 \\ Q_{20}(t) & 0 & 0 & 0 & Q_{24}(t) \\ 0 & Q_{31}(t) & Q_{32}(t) & 0 & 0 \\ 0 & Q_{41}(t) & Q_{42}(t) & 0 & 0 \end{bmatrix},$$

where

$$Q_{01}(t) = a_1 G(t), \quad Q_{02}(t) = a_2 G(t),$$

$$Q_{10}(t) = \int_0^t [1 - U_1(t)] dF_1(x),$$

$$Q_{13}(t) = \int_0^t [1 - F_1(t)] dU_1(x),$$

$$Q_{20}(t) = \int_0^t [1 - U_2(t)] dF_2(x),$$

$$Q_{24}(t) = \int_0^t [1 - F_2(t)] dU_2(x),$$

$$Q_{31}(t) = a_1 V_1(t), \quad Q_{32}(t) = a_2 V_1(t)$$

$$Q_{41}(t) = a_1 V_2(t), \quad Q_{42}(t) = a_2 V_2(t)$$

From *Definition 1* it follows that this process can be treated as a perturbed process with respect to a semi-Markov process $\{X^0(t): t \geq 0\}$ with a state space $A = \{1, 2, 3, 4\}$ and a kernel

$$\mathrm{Q}^0(t) = \begin{bmatrix} 0 & 0 & Q^0_{13}(t) & 0 \\ 0 & 0 & 0 & Q^0_{24}(t) \\ Q^0_{31}(t) & Q^0_{32}(t) & 0 & 0 \\ Q^0_{41}(t) & Q^0_{42}(t) & 0 & 0 \end{bmatrix},$$

where

$$Q^0_{13}(t) = U_1(t), \quad Q^0_{24}(t) = U_2(t),$$

$$Q^0_{31}(t) = a_1 V_1(t), \quad Q^0_{32}(t) = a_2 V_1(t),$$

$$Q^0_{41}(t) = a_1 V_2(t), \quad Q^0_{42}(t) = a_2 V_2(t).$$

The transition probability matrix of an embedded Markov chain of semi-Markov process $\{X^0(t): t \geq 0\}$ have the form

$$\boldsymbol{P}^0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_1 & a_2 & 0 & 0 \\ a_1 & a_2 & 0 & 0 \end{bmatrix}.$$

The stationary distribution $\boldsymbol{\pi}^0 = [\pi^0_1, \pi^0_2, \pi^0_3, \pi^0_4]$ of this Markov chain, we obtain by solving the linear equation system

$$\boldsymbol{\pi}^0 \boldsymbol{P}^0 = \boldsymbol{\pi}^0, \quad \sum_{i=0}^{4} \pi^0_i = 1.$$

The solution is $\pi^0 = \left[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right]$.

The other parameters from *Theorem 1* that we need

$$\varepsilon_1 = \int_0^\infty [1 - U_1(x)] dF_1(x),$$

$$\varepsilon_2 = \int_0^\infty [1 - U_2(x)] dF_2(x),$$

$$\varepsilon_3 = 0,$$

$$\varepsilon_4 = 0,$$

$$m_1^0 = E(U_1) = \int_0^\infty [1 - U_1(t)] dt,$$

$$m_2^0 = E(U_2) = \int_0^\infty [1 - U_2(t)] dt,$$

$$m_3^0 = E(V_1) = \int_0^\infty [1 - V_1(t)] dt,$$

$$m_4^0 = E(V_2) = \int_0^\infty [1 - V_2(t)] dt,$$

$$\varepsilon = \sum_{i=1}^4 \pi_i^0 \varepsilon_i$$

$$= \frac{1}{4}\left(\int_0^\infty [1 - U_1(x)] dF_1(x) + \int_0^\infty [1 - U_2(x)] dF_2(x)\right),$$

$$\lambda = \frac{4}{E(U_1) + E(U_2) + E(V_1) + E(V_2)}.$$

Finally we obtain the approximate reliability function

$$R(t) \approx \left[\frac{\int_0^\infty [1 - U_1(x)] dF_1(x) + \int_0^\infty [1 - U_2(x)] dF_2(x)}{E(U_1) + E(U_2) + E(V_1) + E(V_2)} t\right]. \quad (18)$$

4. Pavlov and Ushakov concept of the perturbed semi-Markov process

We introduce Pavlov and Ushakov [10] concept of the perturbed semi-Markov process presented by [5].

Let $A' = S - A$ be a finite subset of states and A be at least countable subset of S. Suppose that $\{X(t) : t \geq 0\}$ is SM process with the state space $S = A \cup A'$ and the kernel $\boldsymbol{Q}(t) = [Q_{ij}(t) : i, j \in S]$, the elements of which have the form $Q_{ij}(t) = p_{ij} F_{ij}(t)$.

Assume that

$$\varepsilon_i = \sum_{j \in A} p_{ij} \quad (19)$$

and

$$p_{ij}^0 = \frac{p_{ij}}{1 - \varepsilon_i}, \quad i, j \in A'. \quad (20)$$

Let us notice that $\sum_{j \in A'} p_{ij}^0 = 1$.

A semi-Markov process $\{X(t) : t \geq 0\}$ with the discrete state space S defined by the renewal kernel $\mathrm{Q}(t) = [p_{ij} F_{ij}(t) : i, j \in S]$, is called the perturbed process with respect to SM process $\{X^0(t) : t \geq 0\}$ with the state space A' defined by the kernel

$$\boldsymbol{Q}^0(t) = [p_{ij}^0 F_{ij}(t) : i, j \in A'].$$

We are going to present the theorem that is our version of theorem proved by [5].

The number

$$m_i^0 = \int_0^\infty [1 - G_i^0(t)] dt, \quad i \in A', \quad (21)$$

where

$$G_i^0(t) = \sum_{j \in A'} Q_{ij}^0(t) \quad (22)$$

is the expected value of the waiting time in state i for the process $\{X^0(t) : t \geq 0\}$. Denote the stationary distribution of the embedded Markov chain in SM process $\{X^0(t) : t \geq 0\}$ by $\pi^0 = [\pi_i^0 : i \in A']$.

Let

$$\varepsilon = \sum_{i \in A'} \pi_i^0 \varepsilon_i, \quad m^0 = \sum_{i \in A} \pi_i^0 m_i^0. \quad (23)$$

We are interested in the limiting distribution of the random variable Θ_{iA} denoting the first passage time from the state $i \in A'$ to the subset A.

Theorem 2. [10] If the embedded Markov chain defined by the matrix of transition probabilities

$\boldsymbol{P} = [p_{ij} : i, j \in S]$, satisfies the following conditions

$$f_{iA} = P(\Delta_A < \infty \mid X(0) = i) = 1, \quad i \in A', \tag{24}$$

$$\bigvee_{c>0} \bigwedge_{i,j \in S} 0 < E(T_{ij}) \le c, \tag{25}$$

$$\bigwedge_{i \in A} \mu_{iA} = \sum_{n=1}^{\infty} n f_{iA}(n) < \infty, \tag{26}$$

then

$$\lim_{\varepsilon \to 0} P(\varepsilon \Theta_{iA} > x) = e^{-\frac{x}{m^0}}, \tag{27}$$

where $\pi^0 = [\pi_i : i \in A']$ is the unique solution of the linear system of equations

$$\pi^0 = \pi^0 \boldsymbol{P}^0, \quad \pi^0 1 = 1. \tag{28}$$

From that theorem it follows that for small ε we get the following approximating formula

$$P(\Theta_{iA} > t) \approx \exp\left[-\frac{\sum_{i \in A'} \pi_i^0 \varepsilon_i}{\sum_{i \in A'} \pi_i^0 m_i^0} t \right], \quad t \ge 0. \tag{29}$$

4. 1. Reliability of a cold standby system

The presented model is some modification of the model that was considered by [1], [3] and many others. To describe the reliability evolution of the system, we construct a Semi-Markov process by defining the states and the renewal kernel of that one. In our model the time to failure of the system is represented by a random variable that denotes the first passage time from the given state to the subset of states. Appropriate theorems from the Semi-Markov processes theory allow us to calculate the reliability function and mean time to failure of the system. As calculating an exact reliability function of the system by using Laplace transform is often complicated matter we obtain an approximate reliability function applying a *Theorem 2* concerning the perturbed Semi-Markov processes.

4.2. Description and assumptions

We assume that the system consists of one operating series unit, an identical stand- by unit (component) and a switch. We assume that time to failure of both units are represented by non-negative mutually independent copies of a random variable ς with distributions given by probability density functions (pdf) $f(x), x \ge 0$. When the operating component fails, the spare is put in motion by the switch immediately. The failed is renewed. There is a single repair facility. A renewal time is a random variable having a distribution depending on a failed component. We suppose that the lengths of repair periods of units are represented by identical copies of non-negative random variables γ which have cumulative distribution functions $H(x) = P(\gamma \le x), x \ge 0$. The failure of the system occurs when the operating unit fails and the component that has sooner failed in not still renewed or when the operating component fails and the switch also fails. Let U be a random variable having a binary distribution

$$b(k) = P(U = k) = a^k (1-a)^{1-k}, \; k = 0,1, \; a \in (0,1),$$

where $U = 0$, if if a switch is failed at the moment of the operating unit failure, and $U = 1$, if the switch works at that moment. We suppose that the whole failed system is replaced by the new identical one. The replacing time is a non negative random variable η with CDF

$$K(x) = P(\eta \le x), \; x \ge 0.$$

4.3. Construction of reliability model

To describe the reliability evolution of the system, we have to define the states and the renewal kernel. We introduce the following states:

0 – failure of the system
1 – renewal of the failed component after its failure, a spare unit is working
2 – both an operating unit and a spare are "up".

The schema shown in figure 2 presents functioning of the system. Let $0 = \tau_0^*, \tau_1^*, \tau_2^*, \ldots$ denote the instants of the state changes, and $\{Y(t) : t \ge 0\}$ be a random process with the state space $S = \{0, 1, 2\}$, which keeps constant values on the half-intervals $[\tau_n^*, \tau_{n+1}^*), 0, 1, \ldots$ and is right- hand continuous. This process is not semi- Markov, as no memory property is satisfied for any instants of the state changes of that one.

Let us construct a new random process in a following way. Let $0 = \tau_0$ and $\tau_1, \tau_2, \ldots$ denote instants of the subsystem failures or instants of the whole system renewal. The random process $\{X(t) : t \ge 0\}$ defined by equation

$$X(0)=0,\ X(t)=Y(\tau_n)\ \text{ for } t\in[\tau_n,\tau_{n+1}) \qquad (30)$$

is the semi-Markov one.

To have a semi-Markov process as a model we have to define its initial distribution and all elements of its kernel. From assumption and definition semi-Markov kernel we obtain 3-state semi-Markov process with the kernel

$$\mathrm{Q}(t)=\begin{bmatrix}0 & 0 & Q_{02}(t)\\ Q_{10}(t) & Q_{11}(t) & 0\\ Q_{20}(t) & Q_{21}(t) & 0\end{bmatrix}, \qquad (31)$$

where

$$Q_{02}(t)=K(t), \qquad (32)$$

$$Q_{10}(t)=F(t)-a\int_0^t H(x)dF(x),$$

$$Q_{11}(t)=a\int_0^t H(x)dF(x), \qquad (33)$$

$$Q_{20}(t)=(1-a)F(t), \quad Q_{21}(t)=aF(t).$$

Assume that, the initial state is 2. It means that an initial distribution is

$$\boldsymbol{p}(0)=[\ 0\ \ 0\ \ 1\ ]. \qquad (34)$$

Hence, the semi-Markov model has been constructed.

4.4. Reliability characteristics

In our case the random variable Θ_{iA}, that denotes the first passage time from the state $i=2$ to the subset $A=\{0\}$ represents the time to failure of the system in our model. The function

$$R(t)=P(\Theta_{20}>t)=1-\Phi_{20}(t), \quad t\ge 0 \qquad (35)$$

is the reliability function of the considered cold standby system with repair.

In this case the system of linear equations (28) for the Laplace-Stieltjes transforms with the unknown functions

$$\tilde{\phi}_{i0}(s), \quad t\ge 0, \quad i=1,2,$$

$$\tilde{\phi}_{10}(s)=\tilde{q}_{10}(s)+\tilde{\phi}_{10}(s)\tilde{q}_{11}(s),$$

$$\tilde{\phi}_{20}(s)=\tilde{q}_{20}(s)+\tilde{\phi}_{10}(s)\tilde{q}_{21}(s). \qquad (36)$$

Hence

$$\tilde{\phi}_{10}(s)=\frac{\tilde{q}_{10}(s)}{1-\tilde{q}_{11}(s)},$$

$$\tilde{\phi}_{20}(s)=\tilde{q}_{20}(s)+\frac{\tilde{q}_{21}(s)\tilde{q}_{10}(s)}{1-\tilde{q}_{11}(s)}. \qquad (37)$$

Consequently, we obtain the Laplace transform of the reliability function

$$\tilde{R}(s)=\frac{1-\tilde{\phi}_{20}(s)}{s}. \qquad (38)$$

The transition matrix of the embedded Markov chain of the semi-Markov process $\{X(t): t\ge 0\}$ is

$$\boldsymbol{P}=\begin{bmatrix}0 & 0 & 1\\ p_{10} & p_{11} & 0\\ p_{20} & p_{21} & 0\end{bmatrix}$$

(39)

where

$$p_{10}=1-p_{11},$$

$$p_{11}=P(U=1,\gamma<\varsigma)=a\int_0^\infty H(x)\, fF(x),$$

$$p_{10}=1-a, \quad p_{21}=P(U=1)=a.$$

The CDF of the waiting times T_i, $i=0,1,2$ are

$$G_0(t)=K(t),\ G_1(t)=F(t),\ G_2(t)=F(t).$$

$$E(T_0)=E(\eta),\ E(T_1)=E(\varsigma),\ E(T_3)=E(\varsigma). \qquad (39)$$

In this case the equation (36) takes the form of

$$\begin{bmatrix}1-p_{11} & 0\\ -a & 1\end{bmatrix}\begin{bmatrix}E(\Theta_{10})\\ E(\Theta_{20})\end{bmatrix}=\begin{bmatrix}E(\varsigma)\\ E(\varsigma)\end{bmatrix}. \qquad (40)$$

The solution of it is:

$$E(\Theta_{10}) = \frac{E(\varsigma)}{1-p_{11}}, \tag{41}$$

$$E(\Theta_{20}) = E(\varsigma) + \frac{aE(\varsigma)}{1-p_{11}}. \tag{42}$$

4.5. An approximate reliability function

The considered SM process $\{X(t): t \geq 0\}$ with the state space $S = \{0, 1, 2\}$ we can assume to be the perturbed process with respect to the SM process $\{X^0(t): t \geq 0\}$ with the state space $A' = \{1, 2\}$ and the kernel

$$Q^0(t) = \begin{bmatrix} Q_{11}^0(t) & 0 \\ Q_{21}^0(t) & 0 \end{bmatrix}, \tag{43}$$

where

$$Q_{11}^0(t) = p_{11}^0 F_{11}(t), \quad Q_{21}^0(t) = p_{21}^0 F_{21}(t).$$

Because $A = \{0\}$ and

$$\varepsilon_1 = p_{10} = Q_{10}(\infty) = 1 - a\int_0^\infty G(x) f(x) dx,$$

then

$$p_{11}^0 = \frac{p_{11}}{1-\varepsilon_1} = 1.$$

From

$$\frac{Q_{11}(t)}{p_{11}} = F_{11}(t),$$

we get

$$Q_{11}^0(t) = F_{11}(t) = \frac{\int_0^t G(x) f(x) dx}{\int_0^\infty G(x) f(x) dx}.$$

Notice, that

$$\varepsilon_2 = p_{20} = 1 - a.$$

Hence

$$p_{21}^0 = \frac{p_{21}}{1-\varepsilon_2} = 1.$$

Finally we obtain

$$Q_{21}^0(t) = F_{21}(t) = F(t).$$

The transition matrix of the embedded Markov chain of SM process $\{X^0(t): t \geq 0\}$ is

$$\boldsymbol{P}^0 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}. \tag{44}$$

From the system of equations

$$[\pi_1^0, \pi_2^0] \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = [\pi_1^0, \pi_2^0], \quad \pi_1^0 + \pi_2^0 = 1. \tag{45}$$

we get $\boldsymbol{\pi}^0 = [1, 0]$. It follows from the *Theorem 2* that for a small ε and $t \geq 0$

$$P(\Theta_{iA} > t) \approx \exp\left[-\frac{\sum_{i \in A'} \pi_i^0 \varepsilon_i}{\sum_{i \in A'} \pi_i^0 m_i^0} t \right], \tag{46}$$

where the number

$$m_i^0 = \int_0^\infty [1 - G_i^0(t)] dt, \quad i \in A', \tag{47}$$

is the expected value of the waiting time in state i for the process $\{X^0(t): t \geq 0\}$. Denote the stationary distribution of the embedded Markov chain in SM process $\{X^0(t): t \geq 0\}$ by $\pi^0 = [\pi_i^0 : i \in A']$. Let

$$\varepsilon = \sum_{i \in A'} \pi_i^0 \varepsilon_i, \quad m^0 = \sum_{i \in A} \pi_i^0 m_i^0. \tag{48}$$

Therefore we have

$$\varepsilon = \varepsilon_1 = 1 - a\int_0^\infty H(x) f(x) dx, \tag{49}$$

$$m^0 = m_1^0 = \frac{\int_0^\infty x H(x) f(x) dx}{\int_0^\infty H(x) f(x) dx}. \tag{50}$$

For ε close to 0 we obtain the approximate reliability function of the system

$$R(t) = P(\Theta_{iA} > t) = P(\varepsilon\Theta_{iA} > \varepsilon t) \tag{51}$$

$$\approx \exp\left[-\frac{\varepsilon}{m^0}t\right], \quad t \geq 0.$$

From a shape of the parameter ε it follows that we can apply this formula only if the number $P(\gamma \geq \varsigma)$, denoting a probability of a component failure during a period of an earlier failed component, is small.
Finally we obtain an approximate formula

$$R(t) = P\{\Theta_{20} > t\} \approx \exp\left[-\frac{c(1-ac)}{m_*}t\right], \qquad (52)$$

where

$$c = \int_0^\infty H(x)f(x)dx = P(\gamma < \varsigma), \qquad (53)$$

$$m_* = \int_0^\infty xH(x)f(x)dx.$$

We can conclude that the expectation $E(\Theta_{20})$ denoting the mean time to failure of the considered cold standby system is

$$E(\Theta_{20}) = E(\varsigma) + \frac{aE(\varsigma)}{1-p_{11}},$$

where

$$p_{11} = a\int_0^\infty H(x)dF(x).$$

The cold standby determines the increase of the mean time to failure $1+\frac{a}{1-p_{11}}$ times.
The approximate reliability function of the system is exponential with a parameter

$$\Lambda = \frac{c(1-ac)}{m_*}$$

5. Conclusions

In semi-Markov models the reliability function is ussualy calculated by a cumulative distribution function of a first passage time to subset of the process states. In case of complex semi-Markov models the calculating of the exact probability distribution often is very difficult matter. Then, we may to obtain the approximate reliability function. It is possible by using the results from the theory of semi-Markov processes perturbations. The paper shows how we can do it. Two concepts of the perturbed semi-Markov processes are applied to explain presented method. Shpak's [12] concept of the perturbed semi-Markov process allowed to find the approximate reliability function of a many tasks operation process and Pavlov & Ushakov [10] concept provided possibility of calculation the approximate reliability function of *t* a cold standby system with recover.

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