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Random Failure Rate

Keywords

reliability, random failure rate, semi-Markov process, random-walk, Poisson process, Furry-Yule process

Abstract

The reliability function defined by a failure rate which is a stochastic process with nonnegative and right continuous trajectories is presented in this paper. The reliability function with an at most countable state space semi-Markov failure rate process is investigated. A theorem concerning of equations for a conditional reliability function with a semi-Markov process as a failure rate is presented in this paper. The solutions of the proper renewal equations allow getting the reliability functions for the finite space semi-Markov random walk, for the Poisson process and for the Furry-Yule process as a failure rate.

1. Introduction

We have considered a reliability function of the object under assumptions that the failure rate is a random (stochastic) process with nonnegative and right continuous trajectories. Equations for the conditional reliability functions of an element, under assumptions that the failure rate is a special case of a semi-Markov process or a piecewise Markov process with a finite state space, was introduced by [9]. Kopocińska [10] has considered the reliability of an element with an alternating failure rate. For general semi-Markov process with the finite or countable state space, results from the papers mentioned above were generalized by [3]. The theorem deals with the Markov renewal equations for the conditional reliability function with a general semi-Markov failure rate process was proved by [5]. The solution of an introduced finite linear system of equations for the Laplace transforms allowed obtaining the reliability function for some interesting cases of the semi-Markov failure rate processes.

We should mention that there are many other approaches for a concept of the failure rate. For example: Ouhbi and Limnios [13] define a failure rate function in the semi-Markov systems and they show its nonparametric estimation. Hassett, Dietrich and Szidarovsky [6] present the time-varying failure rates in the availability and reliability analysis of repairable systems, Tanguy C. [15] considers periodic failure rate.

2. Essential concepts of a discrete states and continuous time Semi-Markov process theory

The semi-Markov processes were introduced independently and almost simultaneously by Levy P., Smith W.L., and Takacs L. in 1954-55. The essential developments of the semi-Markov processes theory were proposed by [1], [2], [7], [12], [14].We will present only semi-Markov processes with a discrete state space. Usually a semi-Markov process are constructed by the so called Markov Renewal Chain $\{\xi_n, \vartheta_n : n \in N_0\}$, $\xi_n \in S$, $\vartheta_n \in [0,\infty)$, which is a special case of two-dimensional Markov sequence, such that the transition probabilities depend only on the discrete coordinate

$$P(\xi_{n+1} = j, \vartheta_{n+1} \le t \mid \xi_n = i, \vartheta_n = t_n) =$$

$$= P(\xi_{n+1} = j, \vartheta_{n+1} \le t \mid \xi_n = i)$$

and

$$P(\xi_0 = i, \vartheta_0 = 0) = P(\xi_0 = i).$$

The matrix

$$\mathbf{Q}(t) = [Q_{ij}(t) : i, j \in S], \quad t \ge 0, \tag{1}$$

where

$$Q_{ij}(t) = P(\xi_{n+1} = j, \vartheta_{n+1} \le t \mid \xi_n = i) \qquad (2)$$

is said to be the renewal kernel. Let

$$\tau_0 = 0, \;\; \tau_n := \vartheta_1 + ... +, \vartheta_n. \qquad (3)$$

The stochastic processes $\{v(t): t \ge 0\}$ given by

$$v(t) = n, \text{ for } t \in [\tau_n, \tau_{n+1}), \;\; n \in N_0. \qquad (4)$$

is called counting process. The stochastic process $\{X(t): t \ge 0\}$, defined by the formula

$$X(t) = \xi_n, \text{ for } t \in [\tau_n, \tau_{n+1}), \; n \in N_0. \qquad (5)$$

is said to be the semi-Markov process given by the renewal kernel $Q(t)$.

From the above definition it follows that the semi-Markov processes keep constant values on the half-intervals. From the definition of the semi-Markov process it follows that the sequence $\{X(\tau_n): n = 0, 1, ...\}$ is a homogeneous Markov chain with transition probabilities

$$p_{ij} = P(X(\tau_{n+1}) = j \mid X(\tau_n) = i) = \lim_{t \to \infty} Q_{ij}(t). \qquad (6)$$

The function

$$G_i(t) = P(\tau_{n+1} - \tau_n \le t \mid X(\tau_n) = i) = \sum_{j \in S} Q_{ij}(t). \qquad (7)$$

is a cumulative probability distribution of a random variable T_i that is called a waiting time of the state i. The waiting time T_i is the time spent in state i when the successor state is unknown. The function

$$F_{ij}(t) = P(\tau_{n+1} - \tau_n \le t \mid X(\tau_n) = i, X(\tau_{n+1}) = j) \qquad (8)$$

is a cumulative probability distribution of a random variable T_{ij} that is called a holding time of a state i, if the next state is j. From (6) we have

$$Q_{ij}(t) = p_{ij} F_{ij}(t). \qquad (9)$$

From (9) it follows that a semi- Markov process with a discrete state space can be defined by the transition matrix of the embedded Markov chain $\mathbf{P} = [p_{ij} : i, j \in S]$ and a matrix of CDF of holding times $\mathbf{F}(t) = [F_{ij}(t) : i, j \in S]$. A semi-Markov process $\{X(t): t \ge 0\}$ is said to be regular if the corresponding counting process $\{v(t): t \ge 0\}$ has a finite number of jumps on a finite period with probability 1

$$\forall_{t \in \mathrm{R}_+} P(v(t) < \infty) = 1. \qquad (10)$$

Every semi-Markov process with a finite state space is regular [9].

3. Semi-Markov process as the failure rate

We suppose that the failure rate, denoted by $\{\boldsymbol{\lambda}(t): t \ge 0\}$ is a stochastic process with nonnegative and right continuous trajectories.
An expectation

$$R(\mathrm{t}) = \mathrm{E}\left[\exp\left[-\int_0^{\mathrm{t}} \lambda(\mathrm{u})\mathrm{du}\right]\right] \qquad (11)$$

is said to be a reliability function corresponding to a random failure rate process $\{\boldsymbol{\lambda}(t): t \ge 0\}$ From Fubini's theorem and Jensen's inequality we immediately get the following result.
If

$$\int_0^t E[\lambda(u)]du < \infty \qquad (12)$$

then

$$R(t) \ge \exp\left[-\int_0^t E[\lambda(u)]du\right]. \qquad (13)$$

From above mentioned inequality it follows that the reliability function with the random failure rate $\{\boldsymbol{\lambda}(t): t \ge 0\}$ is greater than or equal to the reliability function with the deterministic failure rate equal to the mean $\bar{\lambda}(t) = E[\lambda(u)]$.

We assume that the failure rate is semi- Markov process taking values on an at most countable state space $S = \{\lambda_j : j \in J\}$, where $J \subset \{0, 1, 2, ...\}$ or $J = \{0, 1, 2, ..., n\}$ (see e.g. [7], [12], [14]).
A conditional expectation

$$R_i(t) = E\left[\exp\left[-\int_0^t \lambda(u)du\right] \mid \lambda(0) = \lambda_i\right] \qquad (14)$$

is said to be a conditional reliability function corresponding to a random failure rate process $\{\boldsymbol{\lambda}(t): t \geq 0\}$ if $\lambda(0) = \lambda_i$.

Theorem 1. [3]
If the failure rate function $\{\boldsymbol{\lambda}(t): t \geq 0\}$ is a regular semi-Markov process on discrete state space $S = \{\lambda_j : j \in J\}$, with a kernel $\mathbf{Q}(t) = [Q_{ij}(t): i, j \in J]$, then the conditional reliability functions $R_i(t)$, $i \in J$, satisfy the system of equations

$$R_i(t) = e^{-\lambda_i t}(1 - G_i(t)) + \int_0^t e^{-\lambda_i x} \sum_{j \in J} R_j(t-x) dQ_{ij}(x), \quad i \in J, \tag{15}$$

where

$$G_i(t) = \sum_{j \in J} Q_{ij}(t). \tag{16}$$

The solution is unique in class of the measurable and uniformly bounded functions.
To solve that system of integral equation we will apply the Laplace transform. Let

$$\tilde{R}_i(s) = \int_0^\infty e^{-st} R_i(t) dt, \quad \tilde{G}_i(s) = \int_0^\infty e^{-st} G_i(t) dt,$$

$$\tilde{q}_{ij}(s) = \int_0^\infty e^{-st} Q_{ij}(t) dt.$$

Passing in (to the Laplace transforms), we obtain the system of linear equations in matrix notation has a form

$$(I - \tilde{q}_\Lambda(s))\tilde{R}(s) = \tilde{H}(s), \tag{17}$$

where

$$\tilde{q}_\Lambda(s) = [\tilde{q}_{ij}(s + \lambda_i): i, j \in J]$$

is the square matrix and

$$[\tilde{R}_i(s): i \in J]',$$

$$[\tilde{H}(s) = \left[\frac{1}{s + \lambda_i} - \tilde{G}_i(s + \lambda_i): i \in J\right]',$$

are one column matrices.

4. The random walk process as the failure rate

Let $\{\boldsymbol{\lambda}(t): t \geq 0\}$ be a semi- Markov process with the state space $S = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ and the kernel

$$\mathbf{Q}(t) = [Q_{ij}(t): i, j = 0, \ldots, n], \tag{18}$$

where

$$Q_{ij}(t) = \begin{cases} a_i G_i(t) & \text{for } j = i-1, i = 1, \ldots, n-1 \\ b_i G_i(t) & \text{for } j = i+1, i = 1, \ldots, n-1 \\ G_0(t) & \text{for } j = 1, i = 0 \\ G_n(t) & \text{for } j = n-1, i = n \\ 0 & \text{otherwise.} \end{cases}$$

The functions $G_0(t)$, $G_1(t)$, $\ldots$, $G_n(t)$ denote the cumulative distribution functions with a nonnegative support $R_+ = [0, \infty)$ and $a_k > 0$, $b_k > 0$, $a_k + b_k = 1$, for $k = 1, \ldots, n-1$. This stochastic process $\{\boldsymbol{\lambda}(t): t \geq 0\}$ is called a semi- Markov random walk or the semi- Markov birth and death process. Suppose that the distributions $G_0(t)$, $G_1(t)$, $\ldots$, $G_n(t)$ are absolutely continuous with respect to the Lebesgue measure. Let $\mathbf{p} = [p_0, p_1, \ldots, p_n]$ be an initial probability distribution of the process. Now, the matrices from the equations (17) are

$$I - \tilde{q}_\Lambda(s) = \begin{bmatrix} 1 & \tilde{d}_0 & 0 & 0 & \cdots & 0 \\ a_1\tilde{d}_1 & 1 & b_1\tilde{d}_1(s) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a_{n-1}\tilde{d}_{n-1} & 1 & b_{n-1}\tilde{d}_{n-1} \\ 0 & 0 & \cdots & 0 & \tilde{d}_n & 1 \end{bmatrix}, \tag{19}$$

where

$$\tilde{g}_i(s) = \int_0^\infty e^{-st} dG_i(t), \quad i = 0, 1, \ldots, n,$$

$$\tilde{d}_i(s) = \tilde{g}_k(s + \lambda_i), \quad i = 0, 1, \ldots, n,$$

$$\tilde{\mathbf{R}}(s)=\begin{bmatrix}\tilde{R}_0(s)\\ \tilde{R}_1(s)\\ \vdots\\ \tilde{R}_n(s)\end{bmatrix},\quad \tilde{\mathbf{H}}(s)=\begin{bmatrix}\frac{1}{s+\lambda_0}-\tilde{G}_0(s+\lambda_0)\\ \frac{1}{s+\lambda_1}-\tilde{G}_1(s+\lambda_1)\\ \vdots\\ \frac{1}{s+\lambda_n}-\tilde{G}_n(s+\lambda_n)\end{bmatrix}.$$

The Laplace transform of the unconditional reliability function is

$$\tilde{\mathbf{R}}(s)=p_0\tilde{R}_0(s)+\ldots+p_n\tilde{R}_n(s). \qquad (20)$$

From (13) it follows that

$$R(t)\geq\bar{R}(t)=\exp\left[-\int_0^t\bar{\lambda}(u)du\right], \qquad (21)$$

where

$$\bar{\lambda}(u)=E[\boldsymbol{\lambda}(t)]=\sum_{k=0}^{n}\lambda_k P_k(t), \qquad (22)$$

$$P_k(t)=P\{\lambda(t)=\lambda_k\},\ \ t\geq 0. \qquad (23)$$

Let

$$P_k=\lim_{t\to\infty}P_k(t),\ \ k\in J. \qquad (24)$$

As a conclusion from theorems presented by Koryoluk and Turbin [11], we obtain a formula

$$P_k=\frac{\pi_k m_k}{\sum_{i=0}^{n}\pi_i m_i},\ \ k=0,\ldots,n, \qquad (25)$$

where

$$m_k=\int_0^\infty[1-G_k(t)]dt,\ \ k=0,\ldots,n, \qquad (26)$$

is an expectation of a waiting time in state λ_k and the stationary probabilities π_k, $k=0,\ldots,n,$ of the embedded Markov chain $\{\lambda(\tau_n):n=0,1,\ldots\}$. satisfy the linear system of equations

$$\sum_{i\in S}\pi_i p_{ij}=\pi_j,\ \ j\in S,\ \ \sum_{j\in S}\pi_j=1 \qquad (27)$$

where

$$p_{ij}=\lim_{t\to\infty}Q_{ij}(t). \qquad (28)$$

The system of equation (27) takes of the form

$$\begin{aligned} a_1\pi_1 &= \pi_0\\ \pi_0+a_2\pi_2 &= \pi_1\\ b_1\pi_1+a_3\pi_3 &= \pi_2\\ \vdots\quad &\ \ \vdots\\ b_{n-1}\pi_{n-1} &= \pi_n\\ \pi_0+\pi_1+\ldots+\pi_n &= 1. \end{aligned} \qquad (29)$$

It is easy to obtain a solution

$$\pi_j=\frac{b_0b_1\ldots b_{j-1}}{a_0a_1\ldots a_j}\pi_0 \quad \text{for } j=1,\ldots,n, \qquad (30)$$

where

$$b_0=1,\ \ a_n=1. \qquad (31)$$

From a condition $\sum_{j=0}^{n}\pi_j=1$ we get

$$\pi_0=\left(1+\sum_{j=0}^{n}\prod_{k=1}^{j}\frac{b_{k-1}}{a_k}\right)^{-1}. \qquad (32)$$

From (25) we obtain a limiting distribution of the semi-Markov random walk with a kernel matrix (18)

$$P_0=\frac{m_0}{m_0+\sum_{j=1}^{n}\left[\prod_{k=0}^{j}\frac{b_{k-1}}{a_k}\right]m_j}, \qquad (33)$$

$$P_j=\frac{\prod_{k=1}^{j}\frac{b_{k-1}}{a_k}m_j}{m_0+\sum_{j=1}^{n}\left[\prod_{k=1}^{j}\frac{b_{k-1}}{a_k}\right]m_j},\ \ j=1,\ldots,n.$$

Finally for large t we obtain an approximate lower bound for the reliability function

$$R(t)\geq\exp\left[-\int_0^t\bar{\lambda}(u)du\right]\approx\exp\left[-\sum_{k=0}^{n}\lambda_k P_k t\right]. \qquad (34)$$

4.1. Alternating process as a failure rate

An alternating process is a special case of a semi-Markov random walk for $n=1$. That kind of random process as a failure rate was discussed by Kopocińska [10]. Now, the failure rate is semi-Markov process $\{\boldsymbol{\lambda}(t): t \geq 0\}$ taking values in the states space $S=\{\lambda_0, \lambda_1\}$, defined by a kernel matrix

$$\mathbf{Q}(t)=\begin{bmatrix} 0 & G_0(t) \\ G_1(t) & 0 \end{bmatrix}, \tag{35}$$

and an initial distribution $\mathbf{p}=[p_0, p_1]$. Now, the matrices from equation (17) take of the forms

$$I-\tilde{q}_\Lambda(s)=\begin{bmatrix} 1 & -\tilde{g}_0(s+\lambda_0) \\ -\tilde{g}_1(s+\lambda_1) & 1 \end{bmatrix},$$

where

$$\tilde{g}_i(s)=\int_0^\infty e^{-st} dG_i(t), \quad i=0,1,$$

$$\tilde{\mathbf{R}}(s)=\begin{bmatrix} \tilde{R}_0(s) \\ \tilde{R}_1(s) \end{bmatrix}, \quad \tilde{\mathbf{H}}(s)=\begin{bmatrix} \frac{1}{s+\lambda_0}-\tilde{G}_0(s+\lambda_0) \\ \frac{1}{s+\lambda_1}-\tilde{G}_1(s+\lambda_1) \end{bmatrix}.$$

A solution of those system of equation is

$$\tilde{R}_0(s)= \\ =\frac{\frac{1}{s+\lambda_0}-\tilde{G}_0(s+\lambda_0)+\tilde{g}_0(s+\lambda_0)\left[\frac{1}{s+\lambda_1}-\tilde{G}_1(s+\lambda_1)\right]}{1-\tilde{g}_0(s+\lambda_0)\tilde{g}_1(s+\lambda_1)}, \tag{36}$$

$$\tilde{R}_1(s)= \\ =\frac{\frac{1}{s+\lambda_1}-\tilde{G}_1(s+\lambda_1)+\tilde{g}_1(s+\lambda_1)\left[\frac{1}{s+\lambda_0}-\tilde{G}_0(s+\lambda_0)\right]}{1-\tilde{g}_0(s+\lambda_0)\tilde{g}_1(s+\lambda_1)}. \tag{37}$$

An unconditional reliability function is

$$\tilde{R}(s)=p_0\tilde{R}_0(s)+p_1\tilde{R}_1(s). \tag{38}$$

Example 1
We assume that an initial distribution and a kernel of the process are

$$\mathbf{p}=[p_0, p_1],$$

$$\mathbf{Q}(t)=\begin{bmatrix} 0 & 1-(1+\beta t)e^{-\beta t} \\ 1-e^{-\alpha t} & 0 \end{bmatrix},$$

$$\alpha>0, \quad \beta>0, \quad t \geq 0.$$

The CDF of waiting times in the states λ_0, λ_1 are

$$G_0(t)=1-(1+\beta t)e^{-\beta t},$$

$$G_1(t)=1-e^{-\alpha t}, \quad t \geq 0.$$

The corresponding Laplace transforms are

$$\tilde{G}_0(s)=\frac{\beta^2}{s(\beta+s)^2}, \quad \tilde{G}_1(s)=\frac{\alpha^2}{s(\alpha+s)^2},$$

$$\tilde{g}_0(s)=\frac{\beta^2}{(\beta+s)^2}, \quad \tilde{g}_1(s)=\frac{\alpha^2}{\alpha+s},$$

We calculate the conditional reliability function for the parameters

$$\lambda_0=0, \quad \lambda_1=0.2, \quad \alpha=0.01, \quad \beta=0.1, \quad p_0=0, \quad p_1=1.$$

Substituting those functions into (37) we get

$$\tilde{R}_1(s)= \\ =\frac{\frac{1}{s+0.2}-\frac{0.01}{(s+0.2)(s+0.21)}+\frac{0.01}{(s+0.21)}\left[\frac{1}{s}-\frac{0.01}{s(s+0.1)^2}\right]}{1-\frac{0.0001}{(s+0.1)^2(s+0.21)}}.$$

Using MATHEMATICA computer system we obtain the reliability function as an inverse of the Laplace transform

$$R_1(t)=1.25e^{-0.2t}-0.495913e^{-0.137016t} \\ +0.245913e^{-0.0729844t}, \quad t \geq 0.$$

The function $R_1(t)$, $t \geq 0$ is equal to unconditional reliability function $R(t)$, $t \geq 0$. A corresponding probability density function is

$$f(t)=0.25e^{-0.2t}-0.067948e^{-0.137016t} \\ +0.0179478e^{-0.0729844t}, \quad t \geq 0.$$

5. Poisson process as the random failure rate

For the Poisson process a following result is obtained.

Theorem 2. [3]
If the random failure rate $\{\boldsymbol{\lambda}(t): t \ge 0\}$ is the Poisson process with parameter $\lambda > 0$, then the reliability function defined as

$$R(t) = E\left[\exp\left[-\int_0^t \lambda(u)du\right]\right],$$

is

$$R(t) = \exp[-\lambda(t - 1 + \exp[-t])]. \quad (39)$$

Let us recall the well known property: if $\{\boldsymbol{\lambda}(t): t \ge 0\}$ is the Poisson process with the parameter $\lambda > 0$, then

$$E[\boldsymbol{\lambda}(t)] = \lambda t, \quad t \ge 0. \quad (40)$$

From inequality (13) we get

$$R(t) \ge \overline{R}(t) = \exp\left[-\frac{\lambda}{2}t^2\right]. \quad (41)$$

The reliability function (39) and the function (41) with $\lambda = 0.2$ are shown in *Figure 1*.

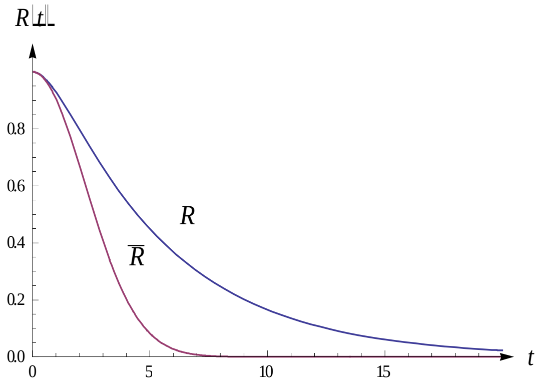

Figure 1. Reliability functions for the Poisson failure rate process.

The density function of the time to failure with the Poisson failure rate is

$$f(t) = \lambda e^{-[\lambda(t-1+e^{-t})]}(1 - e^{-t}). \quad (42)$$

This density function with parameter $\lambda = 0.2$ is shown in *Figure 2*.

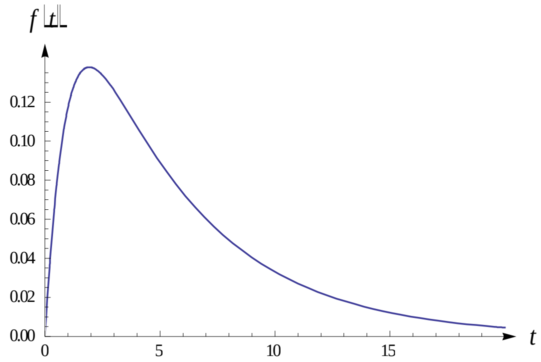

Figure2. Density function for the Poisson failure rate process

The hazard rate function corresponding to the reliability function with the Poisson failure rate we can write as

$$h(t) = \lambda(1 - e^{-t}). \quad (43)$$

Let us notice that $\lim_{t\to\infty} h(t) = \lambda$. It means that for large t the reliability function (39) is approximately equal to the exponential reliability function.

6. Furry-Yule process as the random failure rate

Assume that the random failure rate $\{\boldsymbol{\lambda}(t): t \ge 0\}$ is the Furry-Yule process. The Furry-Yule process with parameter $\lambda > 0$ is the semi-Markov process on the counting state space $S = \{0, 1, 2, ...\}$, defined by the initial distribution $\mathbf{p}(0) = [1, 0, 0, ...]$ and the kernel

$$\mathbf{Q}(t) = \begin{bmatrix} 0 & G_0(t) & 0 & 0 & 0 & \cdots \\ 0 & 0 & G_1(t) & 0 & 0 & \cdots \\ 0 & 0 & 0 & G_2(t) & 0 & \cdots \\ 0 & 0 & 0 & 0 & G_3(t) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad (44)$$

where

$$G_i(t) = 1 - e^{-(i+1)\lambda t}, \; t \ge 0, \; i = 0, 1, \ldots .$$

Theorem 3. [5]
If the random failure rate $\{\boldsymbol{\lambda}(t): t \ge 0\}$ is the Furry-Yule process with parameter $\lambda > 0$, then the reliability function defined as

$$R(t) = E\left[\exp\left[-\int_0^t \lambda(u)du\right]\right],$$

is

$$R(t) = \frac{(\lambda+1)e^{-\lambda t}}{1+\lambda e^{-(\lambda+1)t}}. \tag{45}$$

Well known equalities for the differentiable reliability function come to conclusion:
a density function $f(t)$, $t \geq 0$, and a hazard rate function $h(t)$, $t \geq 0$, corresponding to the reliability function (45) are

$$f(t) = \frac{\lambda(\lambda+1)e^{-\lambda t}(1-e^{-(\lambda+1)t})}{(1+\lambda e^{-(\lambda+1)t})^2}, \tag{46}$$

$$h(t) = \frac{\lambda[1-e^{-(\lambda+1)t}]}{1+\lambda e^{-(\lambda+1)t}}. \tag{47}$$

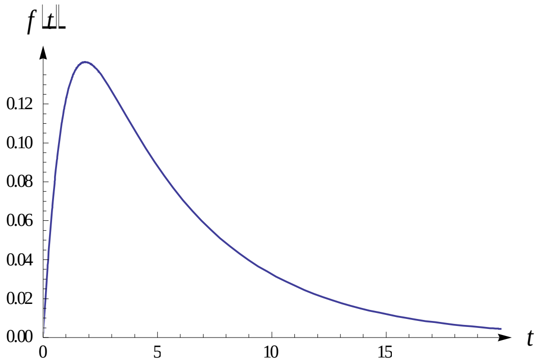

Figure 3. Density function for the Furry- Yule failure rate process.

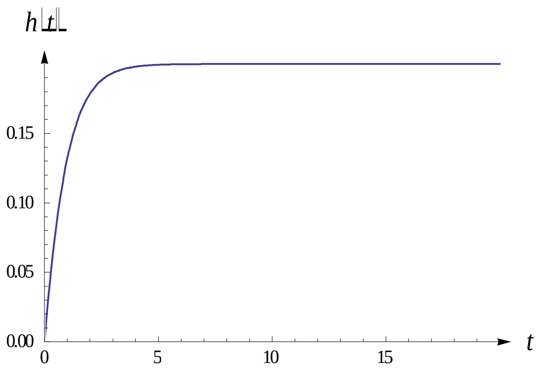

Figure 4. Hazard rate function for the Furry-Yule failure rate process.

Figure 3 shows the density function (46) for $\lambda = 0.2$ and *Figure 4* shows the corresponding hazard rate function . Let us notice that $h(0) = 0$ and $\lim_{t\to\infty} h(t) = \lambda$. An expectation of the Furry- Yule failure rate process is a function $\bar{\lambda}(t) = E[\boldsymbol{\lambda}(t)] = e^{\lambda t} - 1$. Hence the lower bound of the reliability function (45) is

$$\bar{R}(t) = e^{-\frac{1}{\lambda}e^{\lambda t}+\frac{1}{\lambda}+t}. \tag{48}$$

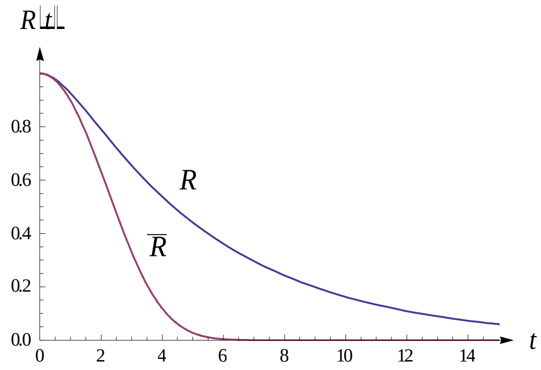

Figure 5. Reliability function and its lower bound for the Furry-Yule failure rate process

7. Conclusion

The randomly changeable environmental conditions cause random load of an object and it implies the random failure rate of that one. For the reliability function defined by a random failure rate we obtained an interesting property: the reliability function with the random failure rate is greater than or equal to the reliability function with the deterministic failure rate equal to the mean of the corresponding random failure rate. A main discussed problem is the reliability function defined by the semi-Markov failure rate process. For the semi-Markov failure rate we have derived equations that allow us to obtain the conditional reliability functions. Applying the Laplace transformation for the introduced system of the renewal equations for an at most countable states space, we have obtained the reliability function for the special cases of the semi-Markov random walk, for the Poisson and Furry-Yule processes as the failure rates. Moreover, we have derived the lower bounds for the considered reliability functions. It seems to be possible to extend presented results on the continuous time non-homogeneous semi-Markov process.

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