Kołowrocki Krzysztof

Soszyńska-Budny Joanna
Maritime University, Gdynia, Poland

Preliminary approach to safety analysis of critical infrastructures

Keywords

safety, multistate system, aging, operation process, dependability, critical infrastructure

Abstract

In the paper the new results of the safety investigations of the multistate complex systems with dependent components at variable operation conditions called critical infrastructures are presented. The multi-state safety function of the critical infrastructure system is defined and determined for an exemplary critical infrastructure. In the developed models, it is assumed that the system components have the multistate exponential safety functions with interdependent departures rates from the subsets of the safety states.

1. Introduction

Currently, the newest trends in the safety investigations of complex technical systems analysis are directed to the critical infrastructures. In general, a critical infrastructure is a single complex system of large scale or a network of complex large systems (set of hard or soft structures) that function collaboratively and synergistically in order to ensure to a continuous production flow of essentials goods and services. These are complex systems that significant features are inside-system dependencies and outside-system dependencies, that in the case of damage have significantly destructive influence on the health, safety and security, economics and social conditions of large human communities and territory areas. These systems are made of large number of interacting components and even small perturbations can trigger large scale consequences in critical infrastructures that may cause multiple treats in human life and activity. For the above reason, as an extended failure within one of these infrastructures may result in the critical incapacity or destruction and can significantly damage many aspects of human life and further cascading across the critical infrastructure boundaries, they have the potential for multi-infrastructural collapse with unprecedented and transnational dangerous consequences.

Many technical systems belong to the class of complex critical infrastructure systems as a result of the large number of interacting components and subsystems they are built of and their complicated operating processes having significant influence on their safety. This complexity and the inside-infrastructure and outside-infrastructure dependencies and hazards cause that there is a need to develop new comprehensive approaches and general methods of analysis, identification, prediction, improvement and optimization for these complex system safety. We meet complex critical infrastructure systems, for instance, in piping transportation of water, gas, oil and various chemical substances, in port and maritime transportation. Optimization of the structures, operation processes and maintenance strategies of critical infrastructures with respect to their safety and costs is very important and very often also complicated and often not possible to perform by practitioners because of the mathematical complexity of the applied methods. In addition, analyzing the critical infrastructures at their variable operation conditions and considering their changing in time safety structures and their among components and subsystems dependability and resulting in changes of their safety characteristics becomes much more complicated. Adding to this analysis, the outside of the critical infrastructures hazards coming from other systems, from natural cataclysm and from other dangerous events makes the problem essentially more difficult to become solved in order to improve and to ensure high level of these systems safety.

From the point of view of more precise analysis of the safety and effectiveness of critical infrastructures, the developed methods should be based on a multistate approach [5], [10]-[11], [19]-[22] to these complex systems safety analysis instead of normally used two-state approach. This will enable different critical infrastructure inside and outside safety states to be distinguished, such that they ensure a demanded level of the system operation effectiveness with accepted consequences of the dangerous accidents for the environment, population, etc.

In most safety analyses, it is assumed that components of a system are independent. But in reality, especially in the case of critical infrastructures, this assumption is not true, so that the dependencies among the critical infrastructure systems' components and subsystems should be assumed and considered. It is a natural assumption, as after decreasing the safety state by one of components in a subsystem, the inside interactions among the remaining components may cause further components safety states decrease [11], [15]-[16]. In reality, in the critical infrastructures, it may even cause the whole system safety state dangerous degradation.

To tie the results of investigations of the critical infrastructures inside-dependences together with the results coming from the assumed in the critical infrastructures outside-dependencies, the semi-Markov models [1]-[5], [10]-[13] can be used to describe the complex systems operation processes. This linking of the inside and outside the critical infrastructures dependencies and including other outside dangerous events and hazards coming from the environment and from other dangerous processes, under the assumed their structures multi-state models, is the main idea of the critical infrastructures safety analysis methodology. This join considering of all these elements is a main innovative aspect of this approach and the basis for the formulation and development of the new solutions concerned with the modeling, identification, prediction, improvement and optimization of the safety of the complex critical infrastructures related to their operation processes and their inside and outside interactions.

2. Multistate approach to safety analysis

In the multistate safety analysis to define a system composed of $n,\ n \in N$, ageing components we assume that:

- E_i, $i = 1,2,\ldots,n$, are components of a system,
- all components and a system under consideration have the set of safety states $\{0,1,...,z\}$, $z \ge 1$,
- the safety states are ordered, the state 0 is the worst and the state z is the best,
- the component and the system safety states degrade with time t,
- $T_i(u)$, $i = 1,2,\ldots,n$, $n \in N$, are random variables representing the lifetimes of components E_i in the safety state subset $\{u, u+1,..., z\}$, while they were in the safety state z at the moment $t = 0$,
- $T(u)$ is a random variable representing the lifetime of a system in the safety state subset $\{u, u+1,..., z\}$, while it was in the safety state z at the moment $t = 0$,
- $s_i(t)$ is a component E_i safety state at the moment t, $t \in <0,\infty)$, given that it was in the safety state z at the moment $t = 0$,
- $s(t)$ is the system safety state at the moment t, $t \in <0,\infty)$, given that it was in the safety state z at the moment $t = 0$.

The above assumptions mean that the safety states of the ageing system and components may be changed in time only from better to worse, what is shown in *Figure 1*.

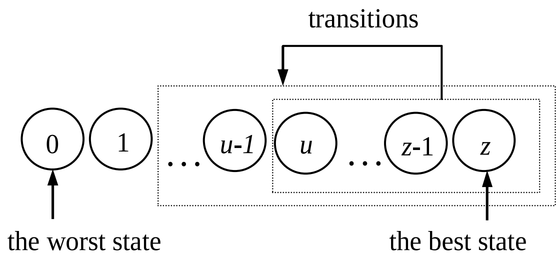

Figure 1. The ageing system and components safety states changes

Definition 1. A vector

$$S_i(t,\cdot) = [S_i(t,0), S_i(t,1), \ldots, S_i(t,z)]$$

for $t \in <0,\infty)$, $i = 1,2,\ldots,n$,

where

$$S_i(t,u) = P(s_i(t) \ge u \mid s_i(0) = z) = P(T_i(u) > t) \quad (1)$$

for $t \in <0,\infty)$, $u = 0,1,\ldots,z$,

is the probability that the component E_i is in the safety state subset $\{u, u+1,..., z\}$ at the moment t, $t \in <0,\infty)$, while it was in the safety state z at the moment $t = 0$, is called the multi-state safety function of a component E_i.

Definition 2. A vector

$$\boldsymbol{S}(t,\cdot) = [\boldsymbol{S}(t,0), \boldsymbol{S}(t,1), \ldots, \boldsymbol{S}(t,z)],\ t \in <0,\infty), \quad (2)$$

where

$$\boldsymbol{S}(t,u) = P(s(t) \geq u \mid s(0) = z) = P(T(u) > t) \qquad (3)$$

for $t \in < 0, \infty)$, $u = 0,1,...,z$,

is the probability that the system is in the safety state subset $\{u, u+1, ..., z\}$ at the moment t, $t \in < 0, \infty)$, while it was in the safety state z at the moment $t = 0$, is called the multi-state safety function of a system.

It is clear that from *Definition 1* and *Definition 2*, for $u = 0$, we have $S_i(t,0) = 1$ and $\boldsymbol{S}(t,0) = 1$.

The safety functions $\boldsymbol{S}(t,u)$, $t \in < 0, \infty)$, $u = 0,1,...,z$, defined by (3) are called the coordinates of the system multistate safety function $\boldsymbol{S}(t, \cdot)$ given by (2). Consequently, the relationship between the distribution function $\boldsymbol{F}(t,u)$ of the system S lifetime $T(u)$ in the safety state subset $\{u, u+1, ..., z\}$ and the coordinate $\boldsymbol{S}(t,u)$ of its multistate reliability function is given by

$$\boldsymbol{F}(t,u) = P(T(u) \leq t) = 1 - P(T(u) > t) = 1 - \boldsymbol{S}(t,u),$$

$$t \in < 0, \infty), \ u = 0,1,...,z.$$

Under *Definition 2*, we have

$$\boldsymbol{S}(t,0) \geq \boldsymbol{S}(t,1) \geq \ldots \geq \boldsymbol{S}(t,z), \ t \in < 0, \infty),$$

and if

$$p(t,u) = P(s(t) = u \mid s(0) = z), \ t \in < 0, \infty), \qquad (4)$$

$$u = 0,1,...,z,$$

is the probability that the system is in the safety state u at the moment t, $t \in < 0, \infty)$, while it was in the safety state z at the moment $t = 0$, then

$$\boldsymbol{S}(t,0) = 1, \ \boldsymbol{S}(t,z) = p(t,z), \ t \in < 0, \infty), \qquad (5)$$

and

$$p(t,u) = \boldsymbol{S}(t,u) - \boldsymbol{S}(t,u+1), \ u = 0,1,...,z-1, \qquad (6)$$

$$t \in < 0, \infty).$$

Moreover, if

$$\boldsymbol{S}(t,u) = 1 \text{ for } t \leq 0, \ u = 1,2,...,z,$$

then

$$\mu(u) = \int_0^\infty \boldsymbol{S}(t,u)dt, \ u = 1,2,...,z, \qquad (7)$$

is the mean lifetime of the system in the safety state subset $\{u, u+1, ..., z\}$,

$$\sigma(u) = \sqrt{n(u) - [\mu(u)]^2}, \ u = 1,2,...,z, \qquad (8)$$

where

$$n(u) = 2\int_0^\infty t\, \boldsymbol{S}(t,u)dt, \ u = 1,2,...,z, \qquad (9)$$

is the standard deviation of the system lifetime in the safety state subset $\{u, u+1, ..., z\}$ and moreover

$$\bar{\mu}(u) = \int_0^\infty p(t,u)dt, \ u = 1,2,...,z, \qquad (10)$$

is the mean lifetime of the system in the safety state u while the integrals (7), (9) and (10) are convergent. Additionally, according to (5)-(7) and (10), we get the following relationship

$$\bar{\mu}(u) = \mu(u) - \mu(u+1), \ u = 0,1,...,z-1,$$

$$\bar{\mu}(z) = \mu(z). \qquad (11)$$

Definition 3. A probability

$$\boldsymbol{r}(t) = P(s(t) < r \mid s(0) = z) = P(T(r) \leq t),$$

$$t \in < 0, \infty),$$

that the system is in the subset of safety states worse than the critical safety state r, $r \in \{1,...,z\}$ while it was in the safety state z at the moment $t = 0$ is called a risk function of the multi-state system [10], [14].

Under this definition, from (3), we have

$$\boldsymbol{r}(t) = 1 - P(s(t) \geq r \mid s(0) = z) = 1 - \boldsymbol{S}(t,r), \qquad (12)$$

$$t \in < 0, \infty),$$

and if τ is the moment when the system risk exceeds a permitted level δ, then

$$\tau = \boldsymbol{r}^{-1}(\delta), \qquad (13)$$

where $\boldsymbol{r}^{-1}(t)$, if it exists, is the inverse function of the system risk function $\boldsymbol{r}(t)$.

3. Safety of multistate "*m* out of *n*" system

One of basic multistate safety structures with components aging in time are "*m* out of *n*" systems.
Definition 4. A multi-state system is called "*m* out of *n*" system if its lifetime $T(u)$ in the safety state subset $\{u, u+1,..., z\}$ is given by

$$T(u) = T_{(n-m+1)}(u),\ m = 1,2,...,n,\ u = 1,\dots,z,$$

where $T_{(n-m+1)}(u)$ is the *n*-m+1-th order statistic in the sequence of the component lifetimes $T_1(u)$, $T_2(u)$, . . ., $T_n(u)$.
The above definition means that the multistate "*m* out of *n*" system is in the safety state subset $\{u, u+1,..., z\}$ if and only if at least *m* out of its *n* components are in this safety state subset.
The multistate "*m* out of *n*" system is called a multistate parallel system if $m = 1$, and it is called a multistate series system if $m = n$.
Consequently, the lifetime $T(u)$ in the safety state subset $\{u, u+1,..., z\}$ of the multistate parallel system is given by

$$T(u) = T_{(n)}(u) = \max_{1\le i\le n}\{T_i(u)\},\ u = 1,\dots,z,$$

and the lifetime $T(u)$ in the safety state subset $\{u, u+1,..., z\}$ of the multistate series system is given by

$$T(u) = T_{(1)}(u) = \min_{1\le i\le n}\{T_i(u)\},\ u = 1,\dots,z.$$

Moreover, the multistate parallel system is in the safety state subset $\{u, u+1,..., z\}$ if and only if at least 1 of its *n* components are in this safety state subset and the multistate series system is in the safety state subset $\{u, u+1,..., z\}$ if and only all of its *n* components are in this safety state subset

Definition 5. A multi-state "*m* out of *n*" system is called homogeneous if its component lifetimes $T_i(u)$ in the safety state subset have an identical distribution function

$$F_i(t,\cdot) = [0, F_i(t,1),\dots,F_i(t,z)]$$

for $t \in \langle 0,\infty)$, $i = 1,2,\dots,n$,

with the coordinates

$$F_i(t,u) = F(t,u) \text{ for } t \in \langle 0,\infty),$$

$u = 1,\dots,z$, $i = 1,2,...,n$,

i.e. if its components E_i have the same safety function

$$S_i(t,\cdot) = [1, S_i(t,1),\dots,S_i(t,z)]$$

for $t \in \langle 0,\infty)$, $i = 1,2,\dots,n$,

with the coordinates

$$S_i(t,u) = S(t,u) = 1 - F(t,u)$$

for $t \in \langle 0,\infty)$, $u = 1,\dots,z$, $i = 1,2,...,n$.

Proposition 1. [10] If in a homogeneous multi-state "*m* out of *n*" system

(i) the components have exponential safety function given by

$$S_i(t,\cdot) = [1, S_i(t,1),\dots,S_i(t,z)] \quad (14)$$

for $t \in \langle 0,\infty)$, $i = 1,2,\dots,n$,
where

$$S_i(t,u) = S(t,u) = \begin{cases} 1, & t < 0 \\ \exp[-\lambda(u)t], & \\ t \ge 0, \lambda(u) \ge 0, i = 1,2,...,n & \end{cases} \quad (15)$$

with the intensity of departure $\lambda(u)$ from the safety state subset $\{u, u+1,..., z\}$,

(ii) the components are independent,

then the multistate system safety function is given by the formula

$$\boldsymbol{S}(t,\cdot) = [1, \boldsymbol{S}(t,1),\dots,\boldsymbol{S}(t,z)], \quad (16)$$

where

$$\boldsymbol{S}(t,u) = \sum_{\upsilon=0}^{n-m}\binom{n}{\upsilon}[1-\exp[-\lambda(u)t]]^{\upsilon} \exp[-(n-\upsilon)\lambda(u)t] \quad (17)$$

for $t \ge 0$, $u = 1,\dots,z$.

From *Proposition 1* we obtain the following corollaries.

Corollary 1. If in a homogeneous multi-state parallel system

(i) the components have exponential safety function given by (14)-(15) with the intensity of departure $\lambda(u)$ from the safety state subset $\{u, u+1, ..., z\}$,

(ii) the components are independent,

then the multistate system safety function is given by the formula

$$\boldsymbol{S}(t,\cdot) = [1, \boldsymbol{S}(t,1), \dots, \boldsymbol{S}(t,z)],$$

where

$$\boldsymbol{S}(t,u) = \sum_{\upsilon=0}^{n-1} \binom{n}{\upsilon} [1 - \exp[-\lambda(u)t]]^{\upsilon}$$

$$\exp[-(n-\upsilon)\lambda(u)t]$$

$$= 1 - [1 - \exp[-\lambda(u)t]]^{n} \text{ for } t \geq 0, \ u = 1, \dots, z.$$

Corollary 2. If in a homogeneous multi-state series system

(i) the components have exponential safety function given by (14)-(15) with the intensity of departure $\lambda(u)$ from the safety state subset $\{u, u+1, ..., z\}$,

(ii) the components are independent,

then the multistate system safety function is given by the formula

$$\boldsymbol{S}(t,\cdot) = [1, \boldsymbol{S}(t,1), \dots, \boldsymbol{S}(t,z)],$$

where

$$\boldsymbol{S}(t,u) = \sum_{\upsilon=0}^{0} \binom{n}{\upsilon} [1 - \exp[-\lambda(u)t]]^{\upsilon}$$

$$\exp[-(n-\upsilon)\lambda(u)t]$$

$$= \exp[-n\lambda(u)t] \text{ for } t \geq 0, \ u = 1, \dots, z.$$

4. Safety of multistate "*m* out of *n*" system with dependent components

In a multi-state "*m* out of *n*" system with dependent components we may consider the dependency of the changes of their ageing safety states and assume that after changing the safety state subset by one of the system components to the worse safety state subset, the lifetimes of the remaining system components in this safety state subsets decrease. More exactly, we assume that if anyone of the system component gets out of the safety state subset $\{u, u+1, ..., z\}$, then the safety of the remaining ones is getting worse so that their mean values of the lifetimes $T_i'(u)$ in safety state subset $\{u, u+1, ..., z\}$ become less according to the formula

$$E[T_i'(u)] = E[T_i(u)] - \frac{1}{n} E[T_i(u)] = \frac{n-1}{n} E[T_i(u)],$$

$$i = 1, 2, \dots, n, \ u = 1, 2, \dots, z.$$

Generalizing, if $\upsilon, \upsilon = 0, 1, 2, \dots, n-1$, components of the system is out of the safety state subset $\{u, u+1, ..., z\}$, the mean values of the lifetimes $T_i'(u)$ in the safety state subset $\{u, u+1, ..., z\}$ of the system remaining components are given by

$$E[T_i'(u)] = E[T_i(u)] - \frac{\upsilon}{n} E[T_i(u)] = \frac{n-\upsilon}{n} E[T_i(u)]$$

$$\text{for } i = 1, 2, \dots, n, \ u = 1, 2, \dots, z.$$

Hence, for the case when components have exponential safety functions given by (14)-(15) with the intensity of departure $\lambda(u)$ from the safety state subset $\{u, u+1, ..., z\}$, according to the well known relationship between the lifetime mean value in this safety state subset and the intensity of departure from this safety state subset of the form

$$E[T_i(u)] = \frac{1}{\lambda(u)} \text{ for } i = 1, 2, \dots, n, \ u = 1, 2, \dots, z,$$

we get the following formula for the intensities of departure from this safety state subset of the remaining components

$$\lambda^{(\upsilon)}(u) = \frac{n}{n-\upsilon} \lambda(u) \tag{18}$$

$$\text{for } \upsilon = 0, 1, 2, \dots, n-1, \ u = 1, 2, \dots, z.$$

Proposition 2. [11] If in a homogeneous multi-state "*m* out of *n*" system

(i) the components have exponential safety function given by (14)-(15),

(ii) the components are dependent,

(iii) the intensities of departure from the safety state subsets of the components are given by (18),

then the multistate system safety function is given by the formula

$$\boldsymbol{S}(t,\cdot) = [1, \boldsymbol{S}(t,1), \dots, \boldsymbol{S}(t,z)], \tag{19}$$

where

$$\boldsymbol{S}(t,u) = \sum_{\upsilon=0}^{n-m} \frac{[n\lambda(u)t]^{\upsilon}}{\upsilon!} \exp[-n\lambda(u)t] \qquad (20)$$

for $t \ge 0$, $u = 1,\dots,z$.

Proof: We denote by $N(t,u)$, the number of components that got out of the safety state subset $\{u,u+1,\dots,z\}$ up to the moment t, $t \ge 0$. Then the number $N(t,u)$ is a process with the states $0,1,2,\dots,n$ and the probability of its particular state is given by

$$P_{\upsilon}(t,u) = P(N(t,u) = \upsilon)$$

for $\upsilon = 0,1,2,\dots,n$, $u = 1,\dots,z$.

The above definition and the formula (18) mean that the process $N(t,u)$ is a Markov process with the transition rates

$$\lambda_{ij}(u) = \begin{cases} (n-i)\dfrac{n}{n-i}\lambda(u) \\ = n\lambda(u), \quad j = i+1 \\ 0, \qquad , j \ne i+1, i = 0,1,\dots,n, \end{cases}$$

i.e. with the following matrix of the transition rates between the states.

$$\begin{array}{c} \text{state} \\ 0 \\ 1 \\ \vdots \\ n-m \\ n-m+1 \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{ccccccccc} 0 & 1 & 2 & \dots & n-m & n-\mathrm{m}+1 & \dots & n & \end{array} \\ \begin{bmatrix} -n\lambda(u) & n\lambda(u) & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & -n\lambda(u) & n\lambda(u) & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -n\lambda(u) & n\lambda(u) & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & -n\lambda(u) & n\lambda(u) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & -n\lambda(u) \end{bmatrix}_{(n+1)\times(n+1).} \end{array}$$

Since the "m out of n" system is out of the safety state subset $\{u,u+1,\dots,z\}$ if at least m its components is in the reliability state subset $\{u,u+1,\dots,z\}$, i.e. the system is out of this safety state subset when the process $N(t,u)$ is at the state $n-m+1$, then the state $n-m+1$ is the absorbing state and the above matrix of transition rates takes the following form

$$\begin{array}{c|ccccccccc|}
\text{state} & 0 & 1 & 2 & \ldots & n-m & n-m+1 & \ldots & & n \\
0 & -n\lambda(u) & n\lambda(u) & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
1 & 0 & -n\lambda(u) & n\lambda(u) & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
n-m & 0 & 0 & 0 & \ldots & -n\lambda(u) & n\lambda(u) & 0 & \ldots & 0 \\
n-m+1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
n & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0
\end{array}_{(n+1)\times(n+1).} \quad (21)$$

Thus, the distribution function of the system lifetime in the state subset $\{u, u+1, \ldots, z\}$ is given by

$$\boldsymbol{F}(t,u) = P(T(u) < t)$$

$$= P(N(t,u) = n-m+1) = P_{n-m+1}(t,u)$$

for $t \geq 0,\ u = 1, \ldots, z,$

and the suitable safety function component of this system is of the form

$$\boldsymbol{S}(t,u) = 1 - P_{n-m+1}(t,u) \quad (22)$$

for $t \geq 0,\ u = 1, \ldots, z.$

From the stochastic processes theory it follows that having given the matrix (21) of the transition rates between the states, it is possible to find the probability $P_{n-m+1}(t,u)$ from the following system of equations.

$$\begin{cases} P_0'(t,u) = -n\lambda P_0(t,u) \\ P_1'(t,u) = n\lambda P_0(t,u) - n\lambda P_1(t,u) \\ P_2'(t,u) = n\lambda P_1(t,u) - n\lambda P_2(t,u) \\ \vdots \\ P_{n-m}'(t,u) = n\lambda P_{n-m-1}(t,u) - n\lambda P_{n-m}(t,u) \\ P_{n-m+1}'(t,u) = n\lambda P_{n-m}(t,u) \end{cases} \quad (23)$$

for $t \geq 0,\ u = 1, \ldots, z.$

It is supposed that all components of the analyzed system at the beginning are in the state subset $\{u, u+1, \ldots, z\}$, and therefore at the moment $t = 0$ the process $N(t,u)$ $u = 1, \ldots, z,$ is at the state 0. It means that the probability of its particular states $v = 0,1,2, \ldots, n-m+1,$ of the process $N(t,u)$ $u = 1, \ldots, z,$ at the moment $t = 0$ are given by

$$P_0(0,u) = 1,\ P_v(0,u) = 0 \quad (24)$$

for $v = 1,2, \ldots, n-m+1,\ u = 1, \ldots, z.$

Applying Laplace's transform to the system of equations (23) with the initial conditions (24), we obtain

$$\begin{cases} sP_0(s,u) - 1 = -n\lambda P_0(s,u) \\ sP_1(s,u) = n\lambda P_0(s,u) - n\lambda P_1(s,u) \\ \vdots \\ sP_{n-m}(s,u) = n\lambda P_{n-m-1}(s,u) - n\lambda P_{n-m}(s,u) \\ sP_{n-m+1}(s,u) = n\lambda P_{n-m}(s,u), \end{cases}$$

where

$$P_\upsilon(s,u) = \int_0^\infty P_\upsilon(t,u) \exp[-st] dt$$

for $\upsilon = 1,2, \ldots, n-m+1,$

is a transform of the probability $P_\upsilon(t,u)$ of its particular states $\upsilon = 1,2, \ldots, n-m+1.$

Further, for $u = 1, \ldots, z$, we have

$$\begin{cases} P_0(s,u) = \dfrac{1}{s + n\lambda(u)} \\ P_1(s,u) = \dfrac{n\lambda(u)}{s + n\lambda(u)} P_0(s,u) \\ \vdots \\ P_{n-m}(s,u) = \dfrac{n\lambda(u)}{s + n\lambda(u)} P_{n-m-1}(s,u) \\ P_{n-m+1}(s,u) = \dfrac{n\lambda(u)}{s} P_{n-m}(s,u) \end{cases}$$

From this system of equations we conclude that the transforms of the probability $P_{n-m+1}(s,u)$ are equal

$$P_{n-m+1}(s,u) = \frac{[n\lambda(u)]^{n-m+1}}{s[s+n\lambda(u)]^{n-m+1}} \text{ for } u = 1,\dots,z,$$

which may be written in the form of the following series

$$P_{n-m+1}(s,u) = \frac{1}{s} - \sum_{\upsilon=0}^{n-m} \frac{[n\lambda(u)]^{\upsilon}}{[s+n\lambda(u)]^{\upsilon+1}}$$
for $u = 1,\dots,z.$

After application of the inverse Laplace's transform to the above equation, we have

$$P_{n-m+1}(t,u) = 1 - \sum_{\upsilon=0}^{n-m} \frac{[n\lambda(u)t]^{\upsilon}}{\upsilon!} \exp[-n\lambda(u)t]$$
for $t \geq 0,\ u = 1,\dots,z.$

Taking into account the last result and (22) we get (20), which completes the proof. #

From *Proposition 2* we obtain the following corollary.

Corollary 3. If in a homogeneous multi-state "*m* out of *n*" system

(i) the components have exponential safety functions given by (14)-(15),
(ii) the components are dependent;
(iii) the intensities of departures of the components from the safety state subsets are given by (18),

then the time of the system lifetime in the state subset $\{u, u+1,\dots,z\}$ has the multi-state distribution function given by the formula

$$\boldsymbol{F}(t,\cdot) = [0, \boldsymbol{F}(t,1),\dots,\boldsymbol{F}(t,z)], \quad (25)$$

where

$$\boldsymbol{F}(t,u) = 1 - \sum_{\upsilon=0}^{n-m} \frac{[n\lambda(u)t]^{\upsilon}}{\upsilon!} \exp[-n\lambda(u)t] \quad (26)$$
for $t \geq 0,\ u = 1,\dots,z.$

Remark 1. According to the definition of Erlang's distribution, *Corollary 3* means that, the multistate "*m* out of *n*" system lifetime $T(u)$ in the safety state subset $\{u, u+1,\dots,z\}$, $u = 1,2,\dots,z$, can be interpreted as the sum of $n-m+1$ independent random variables with exponential distribution with the intensity of departure from this safety state subset equal to $n\lambda(u)$.

Corollary 4. If in a homogeneous multistate parallel system

(i) the components have exponential safety functions given by (14)-(15),
(ii) the components are dependent;
(iii) the intensities of departures of components from the safety state subsets are given by (18),

then the system safety function is given by the vector

$$\boldsymbol{S}(t,\cdot) = [1, \boldsymbol{S}(t,1),\dots,\boldsymbol{S}(t,z)],$$

where

$$\boldsymbol{S}(t,u) = \sum_{\upsilon=0}^{n-1} \frac{[n\lambda(u)t]^{\upsilon}}{\upsilon!} \exp[-n\lambda(u)t]$$

for $t \geq 0,\ u = 1,\dots,z.$

Corollary 5. If in a homogeneous multistate series system

(i) the components have exponential safety functions given by (14)-(15),
(ii) the components are dependent;
(iii) the intensities of departures of components from the safety state subsets are given by (18),

then the system safety function is given by the vector

$$\boldsymbol{S}(t,\cdot) = [1, \boldsymbol{S}(t,1),\dots,\boldsymbol{S}(t,z)],$$

where

$$\boldsymbol{S}(t,u) = \exp[-n\lambda(u)t] \text{ for } t \geq 0,\ u = 1,\dots,z.$$

5. Safety of multistate "*m* out of *l*"-series system

To define a "*m* out of *l*" – series system, assume that:

– *k* is the number "*m* out of *l*" subsystems of the system,
– *l* is the numbers of components of the "*m* out of *l*" subsystems,
– E_{ij}, $i = 1,2,...,k$, $j = 1,2,...,l$, $k, l \in N$, are components of the system,
– all components E_{ij} have the same safety state set as before $\{0,1,...,z\}$,
– $T_{ij}(u)$, $i = 1,2,...,k$, $j = 1,2,...,l$, $k, l \in N$, are random variables representing the lifetimes of components E_{ij} in the safety state subset $\{u, u+1,..., z\}$, while they were in the safety state z at the moment $t = 0$,

– $s_{ij}(t)$ is a component E_{ij} safety state at the moment t, $t \in <0,\infty)$, while they were in the safety state z at the moment $t = 0$.

Definition 6. A vector

$$S_{ij}(t,\cdot) = [S_{ij}(t,0), S_{ij}(t,1),...,S_{ij}(t,z)], \quad (27)$$

$t \in <0,\infty)$, $i = 1,2,...,k$, $j = 1,2,...,l$,

where

$$S_{ij}(t,u) = P(s_{ij}(t) \geq u \mid s_{ij}(0) = z)$$

$$= P(T_{ij}(u) > t), \quad (28)$$

$t \in <0,\infty)$, $u = 0,1,...,z$,

is the probability that the component E_{ij} is in the safety state subset $\{u, u+1,...,z\}$ at the moment t, $t \in <0,\infty)$, while it was in the safety state z at the moment $t = 0$, is called the multistate safety function of a component E_{ij}.

The safety functions $S_{ij}(t,u)$, $t \in <0,\infty)$, u = 0,1,...,z, defined by (28) are called the coordinates of the component E_{ij}, $i = 1,2,...,k$, $j = 1,2,...,l$, multistate safety function $S_{ij}(t,\cdot)$ given by (27). Thus, the relationship between the distribution function $F_{ij}(t,u)$ of the component E_{ij}, $i = 1,2,...,k$, $j = 1,2,...,l$, lifetime $T_{ij}(u)$ in the safety state subset $\{u, u+1,..., z\}$ and the coordinate $S_{ij}(t,u)$ of its multistate safety function is given by

$$F_{ij}(t,u) = P(T_{ij}(u) \leq t) = 1 - P(T_{ij}(u) > t)$$

$$= 1 - S_{ij}(t,u),\ t \in <0,\infty),\ u = 0,1,...,z.$$

Definition 7. A multistate system is called an "m out of l"-series system if its lifetime $T(u)$ in the safety state subset $\{u, u+1,..., z\}$ is given by

$$T(u) = \min_{1 \leq i \leq k} T_{i(l-m+1)}(u),\ m = 1,2,...,l,\ u = 1,2,...,z,$$

where $T_{i(l-m+1)}(u)$ are the $l-m+1$ order statistics in the set of random variables

$$T_{i1}(u),\ T_{i2}(u),\ ...,\ T_{il}(u),\quad i = 1,2,...,k,\ u = 1,2,...,z.$$

The above definition means that the multi-state "m out of l"-series system is composed of k subsystems that are multi-state "m out of l" systems and it is in the safety state subset $\{u, u+1,..., z\}$ if all its "m out of l" subsystems are in this safety state subset. In this definition l denote the numbers of components in the "m out of l" subsystems. The numbers k, m and l are called the system structure shape parameters.

Definition 8. A multi-state "m out of n"-system is called homogeneous if its components E_{ij} have the same safety function

$$S_{ij}(t,\cdot) = [1, S_{ij}(t,1),\ldots, S_{ij}(t,z)]$$

for $t \in <0,\infty)$, $i = 1,2,\ldots,k$, $j = 1,2,\ldots,l$,

with the coordinates

$$S_{ij}(t,u) = S(t,u) \text{ for } t \in <0,\infty),$$

$$u = 1,2,\ldots,z,\ i = 1,2,\ldots,k,\ j = 1,2,\ldots,l.$$

From *Proposition 1* and *Corollary 2*, we have the following propositions.

Proposition 3. [10] If in a homogeneous regular multi-state "m out of l"-series system

(i) the components have exponential safety function given by

$$S_{ij}(t,\cdot) = [1, S_{ij}(t,1),\ldots, S_{ij}(t,z)] \quad (29)$$

for $t \in <0,\infty)$, $i = 1,2,\ldots,k$, $j = 1,2,\ldots,l$,

where

$$S_{ij}(t,u) = S(t,u) = \begin{cases} 1, & t < 0 \\ \exp[-\lambda(u)t], & \\ t \geq 0, \lambda(u) \geq 0, i = 1,2,...,k, & \\ j = 1,2,...,l & \end{cases} \quad (30)$$

with the intensity $\lambda(u)$ of departure from the safety state subset $\{u, u+1,..., z\}$,

(ii) the components are independent,

then the multistate system safety function is given by the formula

$$\boldsymbol{S}(t,\cdot) = [1, \boldsymbol{S}(t,1),\ldots, \boldsymbol{S}(t,z)], \quad (31)$$

where

$$\boldsymbol{S}(t,u) = [\sum_{\upsilon=0}^{l-m} \binom{l}{\upsilon} [1-\exp[-\lambda(u)t]]^{\upsilon} \quad (32)$$

$\exp[-(l-\upsilon)\lambda(u)t]]^k$

for $t \geq 0, \; u = 1,\ldots,z.$

6. Safety of multistate "*m* out of *l*"-series system with dependent components

We assume similarly as in Section 4 that if $\upsilon, \upsilon = 0,1,2,\ldots,l-1,$ components of each "*m* out of *l*" subsystems of the system is out of the safety state subset $\{u, u+1,..., z\}$, the mean values of the lifetimes $T_{ij}'(u)$ in the safety state subset $\{u, u+1,..., z\}$ of this subsystem remaining components are given by

$$E[T_{ij}'(u)] = E[T_{ij}(u)] - \frac{\upsilon}{l}E[T_{ij}(u)] = \frac{l-\upsilon}{l}E[T_{ij}(u)]$$

for $j = 1,2,\ldots,l, \; u = 1,2,\ldots,z.$

Hence, for the case when subsystem components have exponential safety functions given by (29)-(30) with the intensity of departure $\lambda(u)$ from the safety state subset $\{u, u+1,..., z\}$, according to the well known relationship between the lifetime mean value in this safety state subset and the intensity of departure from this safety state subset of the form

$$E[T_{ij}(u)] = \frac{1}{\lambda(u)} \text{ for } j = 1,2,\ldots,l, \; u = 1,2,\ldots,z,$$

we get the following formula for the intensities of departure from this safety state subset of the subsystem remaining components

$$\lambda^{(\upsilon)}(u) = \frac{l}{l-\upsilon}\lambda(u) \text{ for } \upsilon = 0,1,2,\ldots,l-1, \quad (33)$$
$$u = 1,2,\ldots,z.$$

From *Proposition 2* and *Corollary 3,* we have the following propositions.

Proposition 4. [11]
If in a homogeneous multi-state "*m* out of *l*"-series system
(i) the components have exponential safety function given by (29)-(30),
(ii) the components are dependent,
(iii) the intensities of departure from the safety state subsets of the components are given by (33),

then the multistate system safety function is given by the formula

$$\boldsymbol{S}(t,\cdot) = [1, \boldsymbol{S}(t,1),\ldots,\boldsymbol{S}(t,z)], \quad (34)$$

where

$$\boldsymbol{S}(t,u) = [\sum_{\upsilon=0}^{l-m}\frac{[l\lambda(u)t]^\upsilon}{\upsilon!}\exp[-l\lambda(u)t]]^k \quad (35)$$

for $t \geq 0, \; u = 1,\ldots,z.$

7. Modelling system variable operation conditions

We assume that the system during its operation process is taking $v, v \in N$, different operation states $z_1, z_2, \ldots, z_v$. Further, we define the system operation process $Z(t)$, $t \in <0,+\infty)$, with discrete operation states from the set $\{z_1, z_2, \ldots, z_v\}$. Moreover, we assume that the system operation process $Z(t)$ is a semi-Markov process [1]-[6], [10]-[13] with the conditional sojourn times θ_{bl} at the operation states z_b when its next operation state is z_l, $b, l = 1,2,...,v$, $b \neq l$. Under these assumptions, the system operation process may be described by:
- the vector of the initial probabilities $p_b(0) = P(Z(0) = z_b)$, $b = 1,2,...,v$, of the system operation process $Z(t)$ staying at particular operation states at the moment $t = 0$

$$[p_b(0)]_{1\times v} = [p_1(0), p_2(0),..., p_v(0)]; \quad (36)$$

- the matrix of probabilities p_{bl}, $b, l = 1,2,...,v$, $b \neq l$, of the system operation process $Z(t)$ transitions between the operation states z_b and z_l

$$[p_{bl}]_{v\times v} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1v} \\ p_{21} & p_{22} & \cdots & p_{2v} \\ \cdots \\ p_{v1} & p_{v2} & \cdots & p_{vv} \end{bmatrix}, \quad (37)$$

where by formal agreement

$$p_{bb} = 0 \text{ for } b = 1,2,...,v;$$

- the matrix of conditional distribution functions $H_{bl}(t) = P(\theta_{bl} < t)$, $b, l = 1,2,...,v$, $b \neq l$, of the system operation process $Z(t)$ conditional sojourn times θ_{bl} at the operation states

$$[H_{bl}(t)]_{v\times v} = \begin{bmatrix} H_{11}(t)\; H_{12}(t) \dots H_{1v}(t) \\ H_{21}(t)\; H_{22}(t) \dots H_{2v}(t) \\ \dots \\ H_{v1}(t)\; H_{v2}(t) \dots H_{vv}(t) \end{bmatrix}, \qquad (38)$$

where by formal agreement

$$H_{bb}(t) = 0 \text{ for } b = 1,2,\dots,v.$$

We introduce the matrix of the conditional density functions of the system operation process $Z(t)$ conditional sojourn times θ_{bl} at the operation states corresponding to the conditional distribution functions $H_{bl}(t)$

$$[h_{bl}(t)]_{v\times v} = \begin{bmatrix} h_{11}(t)\; h_{12}(t) \dots h_{1v}(t) \\ h_{21}(t)\; h_{22}(t) \dots h_{2v}(t) \\ \dots \\ h_{v1}(t)\; h_{v2}(t) \dots h_{vv}(t) \end{bmatrix}, \qquad (39)$$

where

$$h_{bl}(t) = \frac{d}{dt}[H_{bl}(t)] \text{ for } b,l = 1,2,\dots,v,\; b \neq l,$$

and by formal agreement

$$h_{bb}(t) = 0 \text{ for } b = 1,2,\dots,v.$$

The mean values of the conditional sojourn times θ_{bl} of the system operation process $Z(t)$ are given by

$$M_{bl} = E[\theta_{bl}] = \int_0^\infty t dH_{bl}(t) = \int_0^\infty t h_{bl}(t)dt, \qquad (40)$$
$$b,l = 1,2,\dots,v,\; b \neq l.$$

From the formula for total probability, it follows that the unconditional distribution functions of the sojourn times θ_b, $b = 1,2,\dots,v$, of the system operation process $Z(t)$ at the operation states z_b, $b = 1,2,\dots,v$, are given by [1]-[6], [10]-[13]

$$H_b(t) = \sum_{l=1}^{v} p_{bl} H_{bl}(t),\; b = 1,2,\dots,v. \qquad (41)$$

Hence, the mean values $E[\theta_b]$ of the system operation process $Z(t)$ unconditional sojourn times θ_b, $b = 1,2,\dots,v$, at the operation states are given by

$$M_b = E[\theta_b] = \sum_{l=1}^{v} p_{bl} M_{bl},\; b = 1,2,\dots,v, \qquad (42)$$

where M_{bl} are defined by the formula (40).
The limit values of the system operation process $Z(t)$ transient probabilities at the particular operation states

$$p_b(t) = P(Z(t) = z_b),\; t \in <0,+\infty),\; b = 1,2,\dots,v,$$

are given by [1]-[6], [10]-[13]

$$p_b = \lim_{t\to\infty} p_b(t) = \frac{\pi_b M_b}{\sum_{l=1}^{v} \pi_l M_l}, \qquad (43)$$
$$b = 1,2,\dots,v,$$

where M_b, $b = 1,2,\dots,v$, are given by (42), while the steady probabilities π_b of the vector $[\pi_b]_{1xv}$ satisfy the system of equations

$$\begin{cases} [\pi_b] = [\pi_b][p_{bl}] \\ \sum_{l=1}^{v} \pi_l = 1. \end{cases} \qquad (44)$$

In the case of a periodic system operation process, the limit transient probabilities p_b, $b = 1,2,\dots,v$, at the operation states defined by (43), are the long term proportions of the system operation process $Z(t)$ sojourn times at the particular operation states z_b, $b = 1,2,\dots,v$.

8. Safety of multistate system at variable operation conditions

We assume that the changes of the system operation process $Z(t)$ states have an influence on the system multistate components E_i, $i = 1,2,\dots,n$, safety and the system safety structure as well. We mark by $T_1^{(b)}(u)$, $T_2^{(b)}(u)$, $\dots, T_n^{(b)}(u)$ the system components $E_1, E_2, \dots, E_n$ conditional lifetimes in the safety states subset $\{u, u+1, \dots, z\}$, $u = 1,2,\dots,z$, and by $T^{(b)}(u)$ the system conditional lifetimes in the safety states subset $\{u, u+1, \dots, z\}$, $u = 1,2,\dots,z$, while the system is at the operation state z_b, $b = 1,2,\dots,v$.

Further, we define the conditional safety function of the system multi-state component E_i, $i = 1,2,...,n,$ while the system is at the operation state z_b, $b = 1,2,...,\nu,$ by the vector [7]-[10], [17]-[20]

$$[S_i(t,\cdot)]^{(b)} = [1, [S_i(t,1)]^{(b)}, ..., [S_i(t,z)]^{(b)}], \qquad (45)$$

where

$$[S_i(t,u)]^{(b)} = P(T_i^{(b)}(u) > t | Z(t) = z_b) \qquad (46)$$

for $t \in <0,\infty)$, $u = 1,2,...,z,$ $b = 1,2,...,\nu,$ and the conditional safety function of the multistate system while the system is at the operation state z_b, $b = 1,2,...,\nu,$ by the vector [7]-[10], [17]-[20]

$$[\boldsymbol{S}(t,\cdot)]^{(b)} = [1, [\boldsymbol{S}(t,1)]^{(b)}, ..., [\boldsymbol{S}(t,z)]^{(b)}, \qquad (47)$$

where

$$[\boldsymbol{S}(t,u)]^{(b)} = P(T^{(b)}(u) > t | Z(t) = z_b) \qquad (48)$$

for $t \in <0,\infty)$, $u = 1,2,...,z,$ $b = 1,2,...,\nu.$
The system conditional lifetimes

$$T^{(b)}(u) = T(T_1^{(b)}(u), T_2^{(b)}(u),...,T_n^{(b)}(u))$$

at the operation states z_b, defined for $u = 1,2,...,z,$ $b = 1,2,...,\nu,$ $n \in N,$ are dependent on the system components conditional lifetimes $T_1^{(b)}(u),$ $T_2^{(b)}(u),$ $...,T_n^{(b)}(u),$ at the operation state z_b and the coordinates of the system conditional multistate safety functions

$$[\boldsymbol{S}(t,u)]^{(b)}$$
$$= [\boldsymbol{S}([S_1(t,u)]^{(b)}, [S_2(t,u)]^{(b)},...,[S_n(t,u)]^{(b)})]^{(b)}$$

at the operation state z_b, defined for $t \in <0,\infty)$, $u = 1,2,...,z,$ $b = 1,2,...,\nu,$ $n \in N,$ are dependent on the components conditional safety function $[S_1(t,u)]^{(b)}$, $[S_2(t,u)]^{(b)}$, $...,[S_n(t,u)]^{(b)}$ at the operation state z_b.
The safety function $[S_i(t,u)]^{(b)}$ is the conditional probability that the component E_i lifetime $T_i^{(b)}(u)$ in the safety state subset $\{u, u+1,..., z\}$ is greater than *t*, while the process *Z*(*t*) is at the operation state z_b.
Similarly, the safety function $[\boldsymbol{s}(t,u)]^{(b)}$ is the conditional probability that the system lifetime $T^{(b)}(u)$ in the safety state subset $\{u, u+1,..., z\}$ is greater than *t*, while the process *Z*(*t*) is at the operation state z_b.
Consequently, we mark by $T(u)$ the system unconditional lifetime in the safety states subset $\{u, u+1,..., z\}$, $u = 1,2,...,z,$ and we define the system unconditional safety function by the vector

$$\boldsymbol{S}(t,\cdot) = [1, \boldsymbol{S}(t,1), ..., \boldsymbol{S}(t,z)], \qquad (49)$$

where

$$\boldsymbol{S}(t,u) = P(T(u) > t) \text{ for } t \in <0,\infty), \qquad (50)$$

$u = 1,2,...,z.$

In the case when the system operation time θ is large enough, the system unconditional safety function is given by [33], [50]

$$\boldsymbol{S}(t,u) \cong \sum_{b=1}^{\nu} p_b [\boldsymbol{S}(t,u)]^{(b)} \qquad (51)$$

for $t \geq 0$, $u = 1,2,...,z,$

where $[\boldsymbol{S}(t,u)]^{(b)}$, $u = 1,2,...,z,$ $b = 1,2,...,\nu,$ are the coordinates of the system conditional safety functions defined by (47)-(48) and p_b, $b = 1,2,...,\nu,$ are the system operation process limit transient probabilities given by (43).

9. Safety of multistate "*m* out of *l*"- series system at variable operation conditions

From *Proposition 1* and *Corollary 2*, we have the following propositions.

Proposition 5. [11] If in a homogeneous regular multi-state "*m* out of *n*"-series system operating at variable conditions

(i) the components have exponential safety function given by

$$[S_{ij}(t,\cdot)]^{(b)} = [1, [S_{ij}(t,1)]^{(b)}, ..., [S_{ij}(t,z)]^{(b)}] \qquad (52)$$

for $t \in <0,\infty)$, $i = 1,2,\ldots,k,$ $j = 1,2,\ldots,l,$

where

$$[S_{ij}(t,u)]^{(b)} = \begin{cases} 1, & t < 0 \\ \exp[-[\lambda(u)]^{(b)} t], & \\ t \geq 0, [\lambda(u)]^{(b)} \geq 0, i = 1,2,...,k, & \\ j = 1,2,...,l & \end{cases} \quad (53)$$

with the intensity of departure $[\lambda(u)]^{(b)}$ from the safety state subset $\{u, u+1,..., z\}$,
(ii) the components are independent,
then the multistate system safety function is given by the formula

$$\boldsymbol{S}(t,\cdot) = [1, \boldsymbol{S}(t,1), \ldots, \boldsymbol{S}(t,z)], \quad (54)$$

where

$$\boldsymbol{S}(t,u) = \sum_{b=1}^{\nu} p_b [\sum_{\upsilon=0}^{l-m} \binom{l}{\upsilon} [1 - \exp[-[\lambda(u)]^{(b)} t]]^{\upsilon}$$

$$\exp[-(l-\upsilon)[\lambda(u)]^{(b)} t]]^k, \quad (55)$$

$$t \geq 0, \ u = 1, \ldots, z.$$

Proposition 6. [11] If in a homogeneous multi-state "*m* out of *n*"-series system
(i) the components have exponential safety function given by (52)-(53) with the intensity $[\lambda(u)]^{(b)}$ of departure from the safety state subset $\{u, u+1,..., z\}$,
(ii) the components are dependent,
(iii) the intensities $[\lambda(u)]^{(b)}$ of departure from the safety state subsets of the components at the operation states z_b are given by (33), i.e.
(iv)

$$[\lambda^{(\upsilon)}(u)]^{(b)} = \frac{l}{l-\upsilon}[\lambda(u)]^{(b)}$$

for $\upsilon = 0,1,2,\ldots, l-1, \ u = 1,2,\ldots, z,$

then the multistate system safety function is given by the formula

$$\boldsymbol{S}(t,\cdot) = [1, \boldsymbol{S}(t,1), \ldots, \boldsymbol{S}(t,z)], \quad (56)$$

where

$$\boldsymbol{S}(t,u) = \sum_{b=1}^{\nu} p_b [\sum_{\upsilon=0}^{l-m} \frac{(l[\lambda(u)]^{(b)} t)^{\upsilon}}{\upsilon!}$$

$$\exp[-l[\lambda(u)]^{(b)} t]]^k \quad (57)$$

for $t \geq 0, \ u = 1, \ldots, z.$

10. Safety of an exemplary critical infrastructure

We consider a parallel-series system composed of components E_{ij}, $i = 1,2,3,$ $j = 1,2,...,36,$ operating at three operation states z_1, z_2 and z_3, i.e. $\nu = 3$. We assume that the system safety structure and the system components' safety characteristics are changing at the various operation states.

At the operation state z_1 the system is composed of one "24 out of 36" subsystem composed of components E_{ij}, $i = 1,$ $j = 1,2,...,36.$ with the safety structure presented in *Figure 2*

.

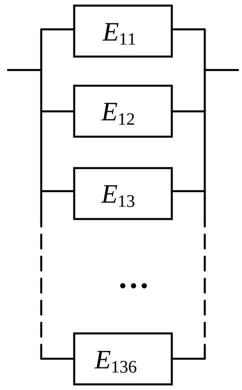

Figure 2. The scheme of the "24 out of 36" system safety structure at the operation state z_1

At the operation state z_2 the system is composed of two "24 out of 36" subsystems linked in series and composed of components E_{ij}, $i = 1,2,$ $j = 1,2,...,36,$ with the safety structure presented in *Figure 3.*

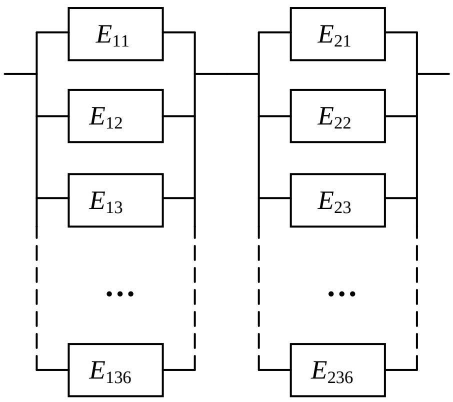

Figure 3. The scheme of the "24 out of 36"-series system safety structure at the operation state z_2

At the operation state z_3, the system is composed of three "24 out of 36" subsystems linked in series and composed of components E_{ij}, $i = 1,2,3,$

$j = 1,2,...,36,$ with the safety structure presented in *Figure 4*.

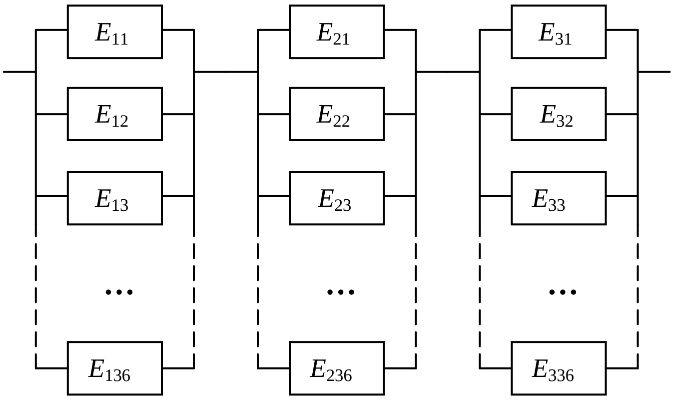

Figure 4. The scheme of the "24 out of 36"-series system safety structure at the operation state z_3

We arbitrarily assume that the transient probabilities of the system at particular operation states z_1, z_2 and z_3 respectively are

$$p_1 = 0.2, \quad p_2 = 0.4, \quad p_3 = 0.4.$$

Moreover, we distinguish four safety states of the system components 0, 1, 2, 3, i.e. $z = 3,$ and we fix that the critical safety state is $r = 2$. Consequently, we define the four-state safety functions of the system components E_{ij}, $i = 1,$ $j = 1,2,...,36,$ at the operation state z_1 in the form of the vector

$$[S_{ij}(t,\cdot)]^{(1)} = [1, [S_{ij}(t,1)]^{(1)}, [S_{ij}(t,2)]^{(1)}, [S_{ij}(t,3)]^{(1)}],$$

$$i = 1, \quad j = 1,2,...,36,$$

with the exponential coordinates

$$[S_{ij}(t,u)]^{(1)} = \exp[-ut], \quad u = 1,2,3 \quad i = 1,$$

$$j = 1,2,...,36,$$

and the four-state safety functions of the system components E_{ij}, $i = 1,2,$ $j = 1,2,...,36,$ at the operation state z_2 in the form of the vector

$$[S_{ij}(t,\cdot)]^{(2)} = [1, [S_{ij}(t,1)]^{(2)}, [S_{ij}(t,2)]^{(2)}, [S_{ij}(t,3)]^{(2)}],$$

$$i = 1,2, \quad j = 1,2,...,36,$$

with the exponential coordinates

$$[S_{ij}(t,u)]^{(2)} = \exp[-2ut], \quad u = 1,2,3, \quad i = 1,2,$$

$$j = 1,2,...,36,$$

and the four-state safety functions of the system components E_{ij}, $i = 1,2,3,$ $j = 1,2,...,36,$ at the operation state z_3 in the form of the vector

$$[S_{ij}(t,\cdot)]^{(3)} = [1, [S_{ij}(t,1)]^{(3)}, [S_{ij}(t,2)]^{(3)}, [S_{ij}(t,3)]^{(3)}],$$

$$i = 1,2,3, \quad j = 1,2,...,36,$$

with the exponential co-ordinates

$$[S_{ij}(t,u)]^{(3)} = \exp[-3ut], \quad u = 1,2,3 \quad i = 1,2,3,$$

$$j = 1,2,...,36.$$

Since the shape parameters of the considered "24 out of 36" system are:

- $k^{(1)} = 1,$ $l^{(1)} = 36,$ $m^{(1)} = 24,$ at the operation state z_1,

- $k^{(2)} = 2,$ $l^{(2)} = 36,$ $m^{(2)} = 24,$ at the operation state z_2,

- $k^{(3)} = 3,$ $l^{(3)} = 36,$ $m^{(3)} = 24,$ at the operation state z_3,

then applying directly the formulae (56)-(57), we get the system safety function

$$\boldsymbol{S}(t,\cdot) = [1, \boldsymbol{S}(t,1), \boldsymbol{S}(t,2), \boldsymbol{S}(t,3)], \quad t \geq 0, \quad (58)$$

where

$$\boldsymbol{S}(t,u)$$

$$\cong \sum_{b=1}^{v} p_b [\sum_{\upsilon=0}^{l^{(b)}-m^{(b)}} \frac{[l^{(b)}\lambda(u)t]^{\upsilon}}{\upsilon!} \exp[-l^{(b)}\lambda(u)t]]^{k^{(b)}}$$

$$= 0.2 \sum_{\upsilon=0}^{12} \frac{[36ut]^{\upsilon}}{\upsilon!} \exp[-36ut]$$

$$+0.4[\sum_{\upsilon=0}^{12}\frac{[72ut]^{\upsilon}}{\upsilon!}\exp[-72ut]]^2$$

$$+0.4[\sum_{\upsilon=0}^{12}\frac{[108ut]^{\upsilon}}{\upsilon!}\exp[-108ut]]^3 \quad (59)$$

for $t \geq 0, u = 1,2,3.$

The approximate graphs of the coordinates of the complex rope system safety function are presented in *Figure 5*.

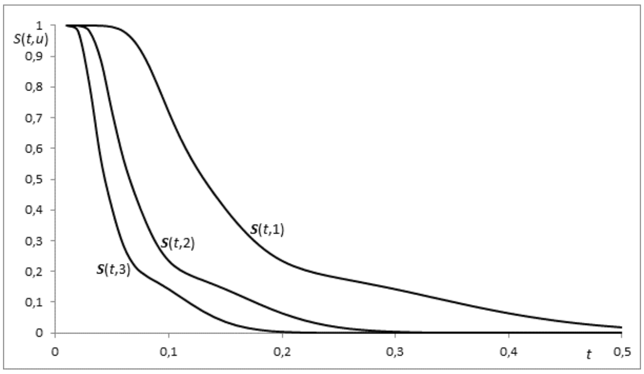

Figure 5. The graph of the exemplary critical infrastructure safety function $\boldsymbol{S}(t,\cdot)$ coordinates

The expected values and standard deviations of the system unconditional lifetimes in the safety state subsets $\{1,2,3\}$, $\{2,3\}$, $\{3\}$, calculated from the results given by (58)-(59), according to (7)-(9), respectively are:

$$\mu(1) \cong 0.169, \ \sigma(1) \cong 0.011, \quad (60)$$

$$\mu(2) \cong 0.085, \sigma(2) \cong 0.003, \quad (61)$$

$$\mu(3) \cong 0.056, \ \sigma(3) \cong 0.001 \quad (62)$$

and further, considering (11) and (60)-(62), the mean values of the unconditional lifetimes in the particular safety states 1, 2, 3, respectively are:

$$\bar{\mu}(1) = \mu(1) - \mu(2) = 0.084$$

$$\bar{\mu}(2) = \mu(2) - \mu(3) = 0.029$$

$$\bar{\mu}(3) = \mu(3) = 0.056. \quad (63)$$

Since the critical reliability state is r = 2, then the system risk function, according to (12), is given by

$$\boldsymbol{r}(t) = 1 - \boldsymbol{S}(t,2)$$

$$= 1 - 0.2\sum_{\upsilon=0}^{12}\frac{[72t]^{\upsilon}}{\upsilon!}\exp[-72t]$$

$$-0.4[\sum_{\upsilon=0}^{12}\frac{[144t]^{\upsilon}}{\upsilon!}\exp[-144t]]^2$$

$$-0.4[\sum_{\upsilon=0}^{12}\frac{[216t]^{\upsilon}}{\upsilon!}\exp[-216t]]^3 \quad (64)$$

Hence, by (13), the moment when the system risk function exceeds a permitted level, for instance δ = 0.05, is

$$\tau = \boldsymbol{r}^{-1}(\delta) \cong 0.066. \quad (65)$$

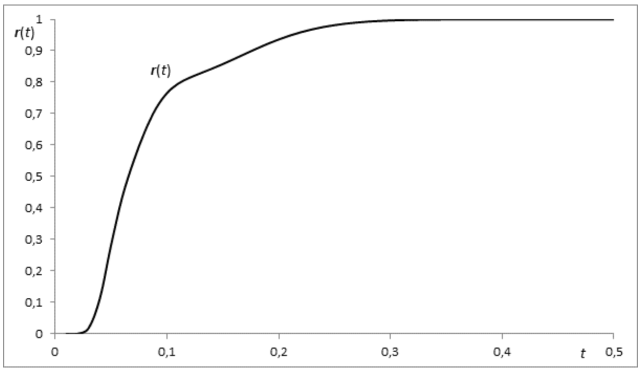

Figure 6. The graph of the exemplary critical infrastructure risk function $\boldsymbol{r}(t)$

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