Journal of Polish Safety and Reliability Association
Summer Safety and Reliability Seminars, Volume 3, Number1, 2012

Hellmich Mario

Berg Heinz-Peter
Bundesamt für Strahlenschutz, Federal Office for Radiation Protection, Salzgitter, Germany

An importance measure for multi-component systems with Semi-Markov dynamics

Keywords

multi-component system, importance measure, Semi-Markov process, state space model

Abstract

We consider semi-Markov reliability models of multi-component systems with a discrete state space, general enough to include systems with maintenance or repair. We assume that for all system states the functioning or failure of each component is specified. In this setup we propose a component importance measure which is close in spirit to the classical steady state Barlow–Proschan importance measure for repairable binary coherent systems. We discuss our importance measure to some extent, highlighting the relation to the classical Barlow–Proschan measure, and present formulas expressing it in terms of quantities easily obtained from the building blocks of the semi-Markov process. Finally an example of a two-component cold standby system with maintenance and repair is presented which illustrates how our importance measure can be used in practical applications.

1. Introduction

For the reliability and safety analysis of complex multi-component systems a number of different quantitative approaches are available. The best known and most frequently applied one is the theory of binary coherent systems ([1] and [2]) which can be regarded as a mathematical sophistication of the traditional fault tree analysis which is widely employed in reliability engineering. However, this method has certain limitations. For example, fault trees cannot model sequence dependent failure modes in which the order of occurrence of events is relevant. Moreover, systems involving complex maintenance and repair operations are hard or even impossible to model using this framework. In order to analyze such systems the so-called state space approaches may be used. Here, the time evolution of the system is modeled by a stochastic process on the set of all possible system states, the so-called state space; in most cases Markov or semi-Markov processes are employed for this purpose. Even though state space models offer a large amount of flexibility to the analyst, they have certain drawbacks. For example, homogeneous semi-Markov models do not cover component aging; moreover, the expenditure to analyze complex systems with a large state space quickly grows. However, state space models are by now well established and widely applied in reliability analysis and engineering as described in [5] to [7] and [17].

Whichever framework to describe complex systems and their time evolutions is used, a number of characteristic parameters can be defined which describe various safety, reliability and performance aspects of the system. For multi-component systems, particularly significant parameters are the so-called component importance measures, which are used to identify those components that are most detrimental to system performance. Since system performance can have many interpretations, such as safety, reliability, availability, etc., it is not surprising that a large number of different importance measures have been developed (see [4] and [9]).

Applications of importance measures include system design, optimization of operation and regulatory purposes, for example in the context of risk informed decision making or in safety reviews. Most of these importance measures are defined and studied in the context of binary coherent systems, and by now there is a copious and mathematically satisfying theory,

which has recently been generalized to include systems with repairable and multistate components in [12] and [13].
However, in the state space approach the problem of constructing and studying importance measures remains largely open. In [16] importance measures for systems with Markov dynamics were studied and explicitly calculated using a perturbation approach, but a systematic study of importance measures is lacking.
In the present paper it is our purpose to report on a component importance measure introduced in [8] for multi-component systems with maintenance and/or component repair, whose time evolution is given by a semi-Markov process. It is close in spirit to the steady state Barlow–Proschan measure for repairable binary coherent systems.
We present results which express the importance measure in terms of quantities easily calculated numerically from the defining quantities of the semi-Markov process, such as transition rates, mean recurrence times, etc., for both the time dependent as well as the steady state case. Moreover, we provide a discussion and an interpretation of the importance measure, and we explain its relation to the classical steady state Barlow–Proschan importance measure for binary coherent systems with component repair.
The underlying mathematical setup which is employed can be briefly described as follows: We consider a system with n components and a finite state space E. The state space is assumed to be partitioned in two disjoint subsets, corresponding to the states of system functioning or failure. Furthermore, we assume that for each system state and each component it is specified whether the component is up (functioning) or down (failed, under repair, on standby, undergoing maintenance, etc.).
As already mentioned above, the time evolution of the system is assumed to be given by a homogeneous semi-Markov process with values in E. Thus, the states successively visited form a Markov chain, and the sojourn time in each state is random, following a distribution which depends on the present state and the state to be visited next.

2. Assumptions and notation

This section introduces our mathematical setup and notation used throughout the paper.

2.1. Multi-component systems

We consider a repairable or maintained system with state space $\mathrm{E} = \{1,...,d\}$. The system is assumed to consist of n components, and we denote the set of components by $\mathscr{C} = \{1,...,n\}$. We suppose that in each system state $i \in E$ a component $a \in \mathscr{C}$ can either be up or down as determined by the maps $c_a : E \to \{0,1\}$, with $c_a(i) = 1$ when a is up in i and $c_a(i) = 0$ when it is down. Moreover, we suppose that the state space is partitioned in two disjoint subsets U and D, i.e. $E = U \cup D$, where U contains all states in which the system is up and D those in which it is down. An example of a system which fits in this framework is given in Section 4.

2.2. Semi-Markov processes

The time evolution of the system under consideration is assumed to be given by a homogeneous semi-Markov process $Z = \{Z(t)\}_{t\geq 0}$ with values in E, defined on some underlying complete probability space $(\Omega, \mathscr{F}, \mathbf{P})$. The corresponding Markov renewal process (MRP) is denoted by $\{(J_n, S_n)\}_{n\in\mathbf{N}}$, which is a sequence of random variables in $E \times \mathbf{R}^+$ such that $0 = S_0 \leq S_1 \leq \ldots$, and such that the following Markov property

$$\mathbf{P}\{J_{n+1} = j, S_{n+1} \leq t \mid J_0,...,J_n, S_0,...,S_n\}$$

$$= \mathbf{P}\{J_{n+1} = j, S_{n+1} \leq t \mid J_n\} = Q_{J_n j}(t - S_n)$$

is satisfied, where $Q_{ij}(t)$ denotes the semi-Markov kernel of the MRP. Upon introducing the random variables $X_0 = S_0 = 0$, $X_n = S_n - S_{n-1}$ and conditioning on $\{J_n = i\}$ the last equation can be written as

$$Q_{ij}(t) = \mathbf{P}\{J_n = j, X_n \leq t \mid J_{n-1} = i\}, \tag{1}$$

independently of n, i.e., the process is homogeneous. The connection between Z and the corresponding MRP is given by $Z(t) = J_{N(t)}$, where the counting process

$$N(t) = \sup\{n \in \mathbf{N} : X_1 + \cdots + X_n \leq t\}$$

counts the number of transitions of Z up to time t. The initial distribution of Z will be denoted by $p_i = \mathbf{P}\{Z(0) = i\}$; moreover, we will write $\mathbf{P}_i\{\cdot\} = \mathbf{P}\{\cdot \mid J_0 = i\}$ as well as $\mathbf{E}_i$ for the corresponding expectation.
The semi-Markov kernel satisfies the following properties:

(i) $Q_{ij}(t) = 0$ for $t \leq 0$,

(ii) the map $t \mapsto Q_{ij}(t)$ is non-decreasing and right-continuous, and

(iii) if $p_{ij} = Q_{ij}(\infty) := \lim_{t\to\infty} Q_{ij}(t)$ then $\sum_{j\in E} p_{ij} = 1$.

In the following, we will occasionally employ the matrix notation $Q(t) = [Q_{ij}(t)]$ and $P = [p_{ij}]$. Conversely, it can be shown that for a matrix function $Q(t)$ satisfying the above three properties (i)–(iii) and for an initial distribution $\{p_i : i \in E\}$ a MRP exists such that (1) is satisfied, hence MRPs, or equivalently semi-Markov processes, can be conveniently constructed by specifying a semi-Markov kernel and an initial distribution. We remark that it can be shown that $\{J_n\}_{n\in\mathbf{N}}$ is a Markov chain with transition matrix P, the so-called embedded Markov chain of the semi-Markov process Z.

We introduce the n-fold matrix convolution power of Q as follows:

$$Q_{ij}^{(0)}(t) = 1_{\{t\geq 0\}}\delta_{ij},$$

$$Q_{ij}^{(1)}(t) = Q_{ij}(t),$$

and

$$Q_{ij}^{(n)}(t) = 1_{\{t\geq 0\}} \sum_{k\in E} \int_0^t Q_{ik}(\mathrm{d}s) Q_{kj}^{(n-1)}(t-s)$$

$$= \mathbf{P}_i\{J_n = j, S_n \leq t\}.$$

The recurrence times of state $j \in E$ are denoted by $S_1^j, S_2^j, \ldots$. Then, the differences $S_{n+1}^j - S_n^j$ are i.i.d. with distribution G_{jj}. Moreover, we denote the distribution of S_1^j conditional on $\{J_0 = i\}$ by G_{ij}. The first moment of G_{jj} is the mean recurrence time μ_{jj} of state j. Let $N_j(t)$ be the process which counts the recurrences of state j, and define the Markov renewal function of Z as follows:

$$\psi_{ij}(t) = \mathbf{E}_i(N_j(t)) = \sum_{n=0}^{\infty} Q_{ij}^{(n)}(t).$$

The matrix $\psi = [\psi_{ij}]$ is called the Markov renewal matrix since it satisfies an integral equation of Markov renewal type.

We finally introduce the transition probabilities of Z as $P_{ij}(t) = \mathbf{P}\{Z(t) = j \mid J_0 = i\}$. Since we will be interested in their limiting behavior as $t \to \infty$ we quote the following result.

Theorem 1: If Z is irreducible and positive recurrent (i.e. the embedded Markov chain is irreducible and positive recurrent), then

$$\lim_{t\to\infty} P_{ij}(t) = \frac{m_j \pi_j}{\sum_{k\in E} m_m \pi_k} =: \omega_j, \tag{2}$$

independently of i, where m_k is the mean sojourn time of the process in state k, and $\{\pi_i : i \in E\}$ is the steady state distribution of the embedded Markov chain, i.e. $\pi_i \geq 0$, $\sum_{i\in E} \pi_i = 1$, and $\pi = [\pi_i]$ is a left eigenvector of P, $\pi P = \pi$.

A more detailed exposition of the theory of semi-Markov processes as well as proofs are provided in [10], [11] and [15].

2.3. Binary coherent systems

The traditional framework for discussing the reliability of complex multi-component systems is the theory of binary coherent systems. In order to exhibit the relation of our framework to the theory of binary coherent systems we introduce some relevant notions here. Again we consider a system consisting of n components, $\mathcal{C} = \{1, \ldots, n\}$, and take its state space to be $E = \{0,1\}^{\times n}$, with the interpretation that $i = (i_1, \ldots, i_n) \in E$ means that component $a \in \mathcal{C}$ is up if $i_a = 1$ and down if $i_a = 0$, thus $c_a(i_1, \ldots, i_n) = i_a$.

The functioning or failure of the system is determined by $U = \{i \in E : \phi(i) = 1\}$ and $D = \{i \in E : \phi(i) = 0\}$, where $\phi : E \to \{0,1\}$ is the structure function of the system, which is (for all practical purposes) equivalent to a coherent fault tree. The structure function is usually required to satisfy two properties: that no component is irrelevant to the system and that improvements of the individual components cannot lead to a degradation of system performance. The time evolution of the system is given by a stochastic process Z with values in E, and we write its coordinates as $Z(t) = (\chi_1(t), \ldots, \chi_n(t)) \in E$, where $\chi_a(t)$ indicates the functioning or failure of component a at time t. We remark that because of component aging and the regenerative property of semi-Markov processes, Z is in general not a semi-Markov process, even if the components are assumed to be independent.

In the framework of binary coherent systems a number of component importance measures have been introduced. The most important one for non-repairable systems is the Birnbaum measure; it is defined by

$$I_{\mathrm{B}}(a,t) = \mathbf{P}\{\phi(1_a, \chi(t)) - \phi(0_a, \chi(t)) = 1\}$$

with $\chi(t) = (\chi_1(t),...,\chi_n(t))$, i.e. it is equal to the probability that the system is in a state at time t in which component a is critical for the functioning of the system. Here we have employed the usual notation

$$(1_a, i) = (i_1,...,i_{a-1},1,i_{a+1},...,i_n),$$

$$(0_a, i) = (i_1,...,i_{a-1},0,i_{a+1},...,i_n),$$

if $i = (i_1,...,i_n) \in E$. A large fraction of the commonly used importance measures for non-repairable binary coherent systems are weighted averages of the Birnbaum measure. For example, if the components are independent the Barlow–Proschan measure is given by

$$I_{\text{B-P}}(a) = \int_0^\infty I_{\mathrm{B}}(a,s) F_a(\mathrm{d}s),$$

where $\overline{F_a(t)} = \mathbf{P}\{\chi_a(t) = 1\}$ is the life distribution of component a, and $\overline{F_a(t)} = 1 - F_a(t)$. Then $I_{\text{B-P}}(a)$ can be interpreted as the probability that component a caused system failure when the system eventually fails. We will come discuss this point in Section 3.3.

3. A Barlow–Proschan type importance measure for Semi-Markov systems

In this section we will introduce our component importance measure for multi-component systems, using the mathematical framework laid out in Section 2.1. The time evolution of the system on E will be assumed to be given by a semi-Markov process Z as introduced in Section 2.2. This setup is general enough to include systems with maintenance and repair.

3.1. Definition and basic properties

Let $i, j \in E$ be two states. We say that a component $a \in \mathcal{C}$ is critical for the transition of the semi-Markov process Z if the following three conditions are satisfied:

C1. On transiting from i to j the system fails, i.e. $i \in U$ and $j \in D$.

C2. On transiting from i to j component a fails, i.e. $c_a(i) - c_a(j) = 1$.

C3. On transiting from i to j no components other than a fail, i.e. $c_b(i) - c_b(j) = 0$ for all $b \in \mathcal{C}$ and $b \neq a$.

The set of all components which are critical for the transition from i to j will be denoted by C_{ij}. Let $N_a(t)$ be the counting process which counts the number of transitions of Z for which component a is critical. We now propose a component importance measure by defining

$$I_{\mathrm{r}}(a,t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}(N_a(t)), \tag{3}$$

provided the derivative exists. We also define a normalized form of $I_{\mathrm{r}}(a,t)$ by

$$I_{\mathrm{r}}^*(a,t) = \frac{I_{\mathrm{r}}(a,t)}{\sum_{b\in\mathcal{C}} I_r(b,t)}, \tag{4}$$

with the property that $\sum_{a\in\mathcal{C}} I_{\mathrm{r}}^*(a,t) = 1$.

Thus, $I_{\mathrm{r}}(a,t)$ is the expected rate of transitions at time t for which component a is critical; the index r indicates the relation of the measure to this transition rate. In other words, $I_r(a,t)\,\mathrm{d}t$ is the probability that system failure together with the failure of component a, but of no component other than a, occurs during the time interval $]t, t+\mathrm{d}t]$. In this sense $I_r(a,t)\,\mathrm{d}t$ is the probability that component a caused system failure during the above time interval. However, in the present framework there is in general no sensible way to define a notion of a causal relationship between system failure and that of a component. Therefore, rather than speaking of a causal relationship, we should more precisely say that a component is associated with system failure.

We can also give an interpretation of the normalized importance measure $I_{\mathrm{r}}^*(a,t)$: Provided that system failure always coincides with the failure of precisely one component,

$$\sum_{b\in\mathcal{C}} I_{\mathrm{r}}^*(b,t)\,\mathrm{d}t$$

is the probability of system failure during the time interval $]t, t+\mathrm{d}t]$.

Consequently, using the definition of conditional probability, $I_{\mathrm{r}}^*(a,t)$ can be interpreted as the probability that component a caused system failure, given that the system fails at time t.

We remark that the notion of component criticality as defined by the above three conditions C1.–C3. should be subjected to critical scrutiny for the concrete application under consideration since, for example, in view of the underlying notion of association of system and component failure, common cause failures which take out more than one component in a transition of the semi-Markov process do not contribute to the importance measure. If this is not tolerable and common cause failures are to be taken into account by the importance measure, condition C3. may be dropped from the list.

3.2. Calculation of the importance measure

After having introduced a component importance measure in Section 3.1. we now present a way to express it explicitly in terms of quantities easily obtainable from the building blocks of the semi-Markov process Z. To this end we quote the following theorem which was proved in [6].

Theorem 2: Suppose that the semi-Markov kernel of Z is absolutely continuous with respect to Lebesgue measure, i.e. $Q_{ij}(\mathrm{d}t) = q_{ij}(t)\mathrm{d}t$, and that

$$\max\{Q_{ij}(t) : i, j \in E\} = \mathcal{O}(t)$$

(this is the case if the density functions q_{ij} are assumed to be continuous at 0 for all $i, j \in E$). Then the derivative in (3) exists and we have

$$I_{\mathrm{r}}(a,t) = \sum_{i,j,k \in E} 1_{\{a \in C_{ij}\}}\, p_k \int_0^t \psi_{ki}(\mathrm{d}s) q_{ij}(t-s). \tag{5}$$

It is worth noting that the proof of this result closely parallels a method from [14].
With the help of (5) we are able to directly calculate $I_{\mathrm{r}}(a,t)$, provided the Markov renewal function ψ of the semi-Markov process is known explicitly. Since usually the semi-Markov process is specified in terms of its semi-Markov kernel, an additional effort is required to find ψ. One possibility is to use the fact that ψ satisfies a Markov renewal equation and to employ the conventional methods of solution, e.g. in terms of Laplace transforms.
However, we can expect a considerable simplification if the semi-Markov process approaches a steady state and the importance $I_{\mathrm{r}}(a,t)$ converges as $t \to \infty$.
This is exemplified by the next result, which is a corollary to Theorem 2.

Corollary 3: Suppose that the assumptions of Theorem 2 are satisfied, and assume in addition that Z is irreducible and recurrent. Moreover, suppose that each function q_{ij}, $i, j \in E$, is direct Riemann integrable. Then

$$I_{\mathrm{r}}^{\mathrm{st}}(a) = \lim_{t\to\infty} I_{\mathrm{r}}(a,t) = \sum_{i,j \in E} 1_{\{a \in C_{ij}\}} \frac{p_{ij}}{\mu_{ii}}, \tag{6}$$

where μ_{ii} is the mean recurrence time of state i.

In this way we have found a time independent importance measure which is not subject to the criticism of time dependent importance measures like $I_{\mathrm{r}}(a,t)$ or $I_{\mathrm{r}}^{*}(a,t)$, in that for them the analyst has to decide at which points of time they are to be evaluated and compared for different components.
In order to evaluate $I_{\mathrm{r}}^{\mathrm{st}}(a)$ explicitly with the help of formula (6) the mean recurrence times μ_{ii} have to be determined. They can be obtained from the following system of linear equations:

$$\mu_{ij} = m_i + \sum_{k \neq j} p_{ik} \mu_{kj},$$

or alternatively from

$$\mu_{ii} = \frac{1}{\pi_i} \sum_{j \in E} \pi_j m_j .$$

As in (4) we also define a normalized version of $I_{\mathrm{r}}^{\mathrm{st}}(a)$ by

$$I_{\mathrm{r}}^{*\mathrm{st}}(a) = \frac{I_{\mathrm{r}}^{\mathrm{st}}(a)}{\sum_{b \in \mathcal{C}} I_{\mathrm{r}}^{\mathrm{st}}(b)},$$

which has the property that $\sum_{a \in \mathcal{C}} I_{\mathrm{r}}^{*\mathrm{st}}(a) = 1$.

Remark 4: In Theorem 2 and Corollary 3 it was assumed that the semi-Markov kernel of Z is absolutely continuous. However, in some applications this assumption may be too restrictive. We can arrive at a generalization of Corollary 3 in the following way. Let $S_1^{ij}, S_2^{ij}, \ldots$ be the times at which a transition from state i to state j occur. Then these times form a renewal process and $S_{n+1}^{ij} - S_n^{ij}$ are i.i.d. with distribution denoted by H_{ij}. Moreover, let $N_{ij}(t)$ be the corresponding counting process. If ν_{ij} denotes the mean of H_{ij} then from the elementary renewal theorem

$$\lim_{t\to\infty}\frac{\mathbf{E}(N_{ij}(t))}{t}=\frac{1}{\nu_{ij}}.$$

Now since

$$N_a(t)=\sum_{i,j\in E}1_{\{a\in C_{ij}\}}N_{ij}(t)$$

we find, using a result from [15] concerning ν_{ij}, that

$$\lim_{t\to\infty}\frac{1}{t}\left|\mathbf{E}(N_a(t))-\sum_{i,j\in E}1_{\{a\in C_{ij}\}}\frac{t}{\nu_{ij}}\right|$$

$$=\lim_{t\to\infty}\frac{1}{t}\left|\mathbf{E}(N_a(t))-\sum_{i,j\in E}1_{\{a\in C_{ij}\}}\frac{tp_{ij}}{\mu_{ii}}\right|.$$

Thus $\mathbf{E}(N_a(t))$ is asymptotically differentiable with derivative still given by (6). In view of this result we can sensibly generalize formula (6) also to semi-Markov processes without any further assumptions about its kernel, and the interpretation of $I_{\mathrm{r}}^{\mathrm{st}}(a)$ given above persists.

3.3. Relation to the classical Barlow–Proschan importance measure for repairable systems

In [3] Barlow and Proschan suggest an importance measure for repairable binary coherent systems, which can be constructed as follows. Consider a binary coherent system of n components as introduced in Section 2.3. We suppose that each component is repaired after failure, and that the failure and repair times for component a are assumed to be distributed according to the distributions F_a and G_a. All failure and repair distributions are assumed to be pairwise independent. If $N_a(t)$ denotes the random variable counting the number of system failures caused by component a until time t, the steady state Barlow–Proschan importance measure for repairable systems is defined by

$$I_{\text{B-P}}^{\mathrm{st}}(a)=\lim_{t\to\infty}\frac{\mathbf{E}(N_a(t))}{\sum_{b\in\mathcal{C}}\mathbf{E}(N_b(t))}. \tag{7}$$

The existence of this limit under the above hypotheses follows from a simple application of the elementary renewal theorem. To arrive at an interpretation of (7) we denote by $M_a(t)$ the number of failures of component a until time t. Then it can be shown that

$$\mathbf{E}(N_a(t))=\int_0^t I_{\mathrm{B}}(a,s)\mathbf{E}(M_a(\mathrm{d}s)).$$

Using this identity and l'Hôpital's rule we can write heuristically

$$I_{\text{B-P}}^{\mathrm{st}}(a)=\lim_{t\to\infty}\frac{I_{\mathrm{B}}(a,t)\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}(M_a(t))}{\sum_{b\in\mathcal{C}}I_{\mathrm{B}}(b,t)\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}(M_b(t))} \tag{8}$$

$$=\lim_{t\to\infty}\frac{I_{\mathrm{B}}(a,t)\,\mathrm{d}\mathbf{E}(M_a(t))}{\sum_{b\in\mathcal{C}}I_{\mathrm{B}}(b,t)\,\mathrm{d}\mathbf{E}(M_b(t))}.$$

Now in view of the interpretation of $I_{\mathrm{B}}(a,t)$ we conclude that $I_{\mathrm{B}}(a,t)\,\mathrm{d}\mathbf{E}(M_a(t))$ is the probability that system failure caused by component a occurs during $]t,t+\mathrm{d}t]$.

Moreover, since the sum over all components of this expression is the probability that system failure occurs during $]t,t+\mathrm{d}t]$ it follows that

$$\frac{I_{\mathrm{B}}(a,t)\,\mathrm{d}\mathbf{E}(M_a(t))}{\sum_{b\in\mathcal{C}}I_{\mathrm{B}}(b,t)\,\mathrm{d}\mathbf{E}(M_b(t))}$$

is the probability that system failure is caused by a, given that the system fails at time t.

Now letting $t\to\infty$ we see that the steady state Barlow–Proschan importance measure can be interpreted as the steady state probability that component a caused system failure, given that system failure has occurred. We finally note that with the help of Blackwell's theorem we can conclude from (8) that

$$I_{\text{B-P}}^{\mathrm{st}}(a)=\frac{I_{\mathrm{B}}(a)/(\lambda_a+\mu_a)}{\sum_{b\in\mathcal{C}}I_{\mathrm{B}}(b)/(\lambda_b+\mu_b)},$$

where $I_{\mathrm{B}}(a)=\lim_{t\to\infty}I_{\mathrm{B}}(a,t)$, and λ_a and μ_a are the expectations of F_a and G_a, respectively.

Consider now an n component system with semi-Markov time evolution as introduced in Section 3.1. As already remarked, if system failure always coincides with the failure of precisely one component, $I_{\mathrm{r}}^{*}(a,t)$ equals the probability that component a caused system failure, given the system fails at time t.

Hence, if we consider a binary coherent system such that the probability of simultaneous failure of two or more components is zero we conclude that the corresponding steady state importance measure

$I_{\mathrm{r}}^{*\mathrm{st}}(a) = \lim_{t\to\infty} I_{\mathrm{r}}^{*}(a,t)$ equals $I_{\mathrm{B\text{-}P}}^{\mathrm{st}}(a)$, provided the limit exists. More formally we can prove:

Proposition 5: Consider a multi-component system with semi-Markov time evolution as in Section 3.1, and suppose that the assumptions of Corollary 3 are satisfied. Then

$$I_{\mathrm{r}}^{*\mathrm{st}}(a) = \lim_{t\to\infty} \frac{\mathbf{E}(N_a(t))}{\sum_{b\in\mathscr{C}} \mathbf{E}(N_b(t))}.$$

The proof is a simple application of l'Hôpital's rule. Thus we see that the steady state importance measure $I_{\mathrm{r}}^{*\mathrm{st}}(a)$ defined by (6) is a generalization of the Barlow–Proschan measure for repairable systems $I_{\mathrm{B\text{-}P}}^{\mathrm{st}}(a)$ as defined in (7).

4. An example of a two-unit cold standby system with maintenance and repair

In this section we present an illustrative example of a two component system with maintenance and repair whose time evolution is given by a semi-Markov process and, hence, fits in the framework laid out in Section 2.1. It serves to explain in which way the component importance measure defined in Section 3.1 can be used in practical applications. The example we present here has appeared repeatedly in the literature with minor variations, for the first time apparently in [1], as well as in [5] and [11].

4.1. Description of the system and its time evolution

The system consists of two components denoted by A and B, i.e. $\mathscr{C} = \{\mathrm{A},\mathrm{B}\}$. The state diagram of the semi-Markov process describing the system's time evolution is pictured in Figure 1.

There are altogether 9 states, i.e. $E = \{1,...,9\}$; the transitions between the states with non-vanishing transition probability are depicted by arrows in Figure 1. Besides being in an "up" condition in which it delivers service, each component can be on standby, indicated by *stby* in Figure 1.

In the standby condition a component delivers no service, but it is assumed that upon failure of the other component it can start up immediately to ensure the functioning of the system.

Preventive maintenance is carried out periodically on both components, indicated by *maint* in Figure 1. During maintenance a component cannot deliver service. If a component has failed it is in the *down* state; after failure it can be put in a repair state as indicated by *rep* in *Figure 1*. From the repair state the component transits to the *stby* state since it is assumed that repair includes maintenance.

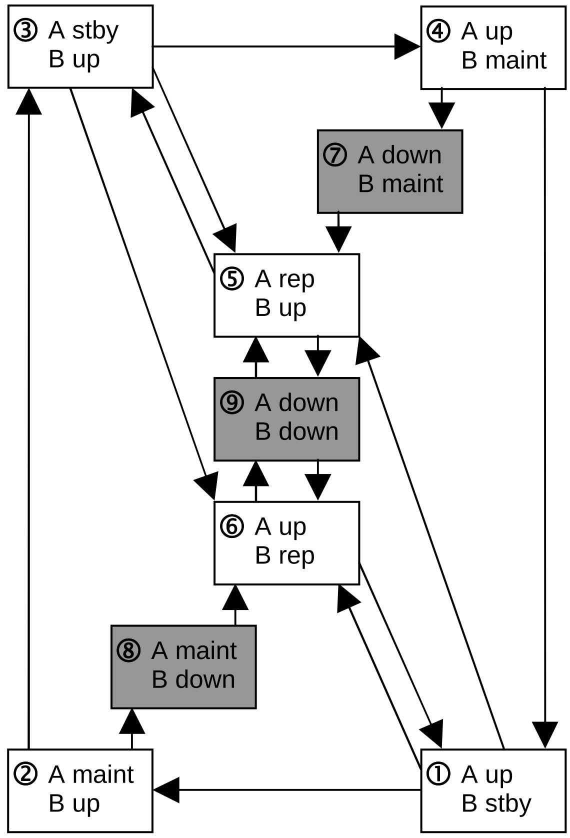

Figure 1. Transition diagram of the semi-Markov process according to [5]. The shaded states are the states of system failure.

The system's states in E can be considered to be pairs of states of each component A and B; they are numbered from 1 to 9 as indicated by the circled numbers in *Figure 1*. The system delivers service if there is at least one component in the *up* state, hence the system is down (i.e. unable to deliver service) in the states 7, 8 and 9 (indicated by shades in *Figure 1*). Thus using the notation of Section 2.1 we have

$$U = \{1, 2, 3, 4, 5, 6\},$$

$$D = \{7, 8, 9\}.$$

Moreover, $c_{\mathrm{A}}(i) = 1$ if $i = 1, 4, 6$, and $c_{\mathrm{A}}(i) = 0$ if $i = 2, 3, 5, 7, 8, 9$, and analogously for component B, i.e. $c_{\mathrm{B}}(i) = 1$ if $i = 2, 3, 5$ and $c_{\mathrm{B}}(i) = 0$ if $i = 1, 4, 6, 7, 8, 9$. In view of the conditions C1.–C3. stated in Section 3.1 we can write the sets of critical transitions C_{A} and C_{B} for the components as follows:

$C_{\mathrm{A}} = \{(4,7), (6,9)\}, \quad C_{\mathrm{B}} = \{(2,8), (5,9)\}$

where the ordered pair (i, j) denotes the transition from state i to state j.

In the following we describe the time evolution of the system in more detail. The components A and B are assumed to be independent with exponential life distributions, with failure rates λ_{A} and λ_{B}. Maintenance is carried out alternately on A and B after c units of time of service and lasts exactly t_0 units of time.

Thus, if no failure occurs the system cycles through states 1, 2, 3 and 4, with a fixed sojourn in each of these states. If component B fails while the system is in state 1, B is repaired and A continues to deliver service (state 6). The repair times are assumed to be independent and exponentially distributed with repair rates μ_{A} and μ_{B}, respectively. If in state 6 component A fails before the repair of component B is completed, the system transits to state 9, yielding system failure. Otherwise the system transits back to state 1. From state 9 transitions to states 5 or 6 are possible, depending on which repair is completed first. Analogously, a transition from state 1 to state 5 occurs if component A fails; then A is repaired and B ensures system operability. If a component fails while the other is in the maintenance state (transitions from 2 to 8 and from 4 to 7), system failure occurs.

It is assumed that the system remains a fixed amount of d units of time in states 7 or 8, respectively, modeling the amount of time necessary to abort maintenance and put the corresponding component to service (transitions to state 5 and 6, respectively). By symmetry, the same considerations apply with states A and B interchanged.

A semi-Markov kernel $Q(t)$ leading to a semi-Markov process modeling the system behavior as described in the previous paragraphs has been given in [5] and [9].

Since this semi-Markov kernel contains components which are not absolutely continuous with respect to Lebesgue Measure (due to the fixed sojourn times given that the process jumps, e.g., from state 1 to state 2), we have to use Remark 4 and (5) to calculate $I_{\mathrm{r}}^{\mathrm{st}}(a)$ and $I_{\mathrm{r}}^{*\mathrm{st}}(a)$.

4.2. Concluding remarks

In the present paper we have reported about a component importance measure for multi-component systems with semi-Markov dynamics, which was introduced in [8].

An illustrative example of a two component system in the semi-Markov framework with maintenance and/or component repair has been given.

So far due to various time constraints of the authors no numerical results for $I_{\mathrm{r}}^{\mathrm{st}}(a)$ or $I_{\mathrm{r}}^{*\mathrm{st}}(a)$ are available, but it is planned to study the dependence of these importance measures on the system parameters such as the failure or repair rates in the context of this example.

Moreover, it is planned to study our importance measures also for examples of further systems beyond the one presented in this section, in particular one which is relevant to nuclear safety.

References

[1] Barlow, R.E. & Proschan, F. (1965). Mathematical Theory of Reliability. SIAM Series in *Appl. Math.*, Wiley, New York.

[2] Barlow, R.E. & Proschan, F. (1975). *Statistical Theory of Reliability and Life Testing. Probability Models*. Holt Rinehart and Winston, Inc., New York.

[3] Barlow, R.E. & Proschan, F. (1975). Importance of system components and fault tree events. *Stoch. Proc. Appl*. 3, 153–173.

[4] Boland, P. J. & El-Neweihi, E. (1995). Measures of component importance in reliability theory. *Comput. Op. Res*. 22, 455–463.

[5] Csenki, A. (1995). An integral equation approach to the interval reliability of systems modeled by finite semi-Markov processes. *Reliab. Eng. Syst. Safety* 47, 37–45.

[6] Csenki, A. (2007) Joint interval reliability for Markov systems with an application in transmission line reliability. *Reliab. Eng. Syst. Safety* 92, 685–696.

[7] Grabski, F. (2010). Semi-Markov reliability model of the cold standby system. *J. Appl. Quant. Meth.* 5, 486–496.

[8] Hellmich, M. & Berg, H.-P. (2012). *Component importance for semi-Markov systems.* Preprint.

[9] Huseby, A. B. (2004). *Importance measures for multicomponent binary systems.* Statist. Res. Report No. 11, Dept. of Math., Univ. of Oslo.

[10] Korolyuk, V.S., Brodi, S.M. & Turbin, A. F. (1975). Semi-Markov processes and their applications. *J. Math. Sci.* 4, 244–280.

[11] Limnios, N. & Oprisan, G. (2001). *Semi-Markov Processes and Reliability.* Birkhäuser, Boston.

[12] Natvig, B. & Gåsemyr, J. (2009). New results on the Barlow–Proschan and Natvig measures of component importance in nonrepairable and repairable systems. *Methodol. Comput. Appl. Probab.* 11, 603–620.

[13] Natvig, B. (2011). Measures of component importance in nonrepairable and repairable multistate systems. *Methodol. Comput. Appl. Probab.* 13, 523–547.

[14] Ouhbi, B. & Limnios, N. (2002). The rate of occurrence of failures for semi-Markov processes and estimation. *Stat. Probab. Lett*. 59, 245–255.

[15] Störmer, H. (1970). *Semi-Markoff-Prozesse mit endlich vielen Zuständen: Theorie und Anwendungen.* Springer Lecture Notes in Operations Research and Mathematical Systems Vol. 34, Berlin, Springer.

[16] Van, P.D. & Barros, A. (2008). Reliability importance analysis of Markovian systems at steady state using perturbation analysis. *Reliab. Eng. Syst. Safety* 93, 1605–1615.

[17] Veeramany, A. & Pandrey, D.P. Reliability analysis of nuclear piping system using semi-Markov process model. *Reliab. Eng. Syst. Safety* 38, 1133–1139.