Zajac Mateusz

Kierzkowski Artur
Wroclaw University of Technology, Wroclaw, Poland

Attempts at calculating chosen contributors with regard to the Semi-Markov process and the Weibull function distribution

Keywords

Semi Markov process, exponential distribution, availability, transient probabilities.

Abstract

The main aim of this article is the presentation of the way of making calculation with the application of the semi-Markov processes, with the continuous time *t*, where non-exponential distribution function is going to be applied. These type of calculations were published but the time there was approaching infinity. Engineering practice proves that the lack of solutions of this issue leads to obtainment of solutions that are encumbered with errors. This article is the first attempt of the error analysis.

1. Introduction

Dependability indices like reliability and related measures, as availability, maintainability, failure rate, mean times, etc., are very important in design, development and lifetime analysis of real systems.
It is worth to point out that there is an assumption that during the calculation of the dependability contributors for technical objects that are under investigation, probabilities of transition between states or sojourn times' probabilities are exponential. Many causes, for example, lack of information, small sample sizes, or inaccurate assessment of data may result in the model assumptions being violated. In some cases, when exponential distribution is assumed, there is also possibility to assess factors according to different distributions, like Weibull, Erlang, etc [11].
Probabilities of transition between states and availability belong to the fundamental characteristic of reliability. The discrete-time case can be obtained from the continuous one, by considering counting measure for discrete time points. However we consider that important is to make it separately for this case, since an increasing interest is observed in practice for the discrete case [1], [2], [7], [8], [10]. The discrete-time model, on one hand, is much simpler to handle numerically than the continuous-time one. On the other hand, it can used to handle numerically continuous-time formulated problems. So for practical reliability problems it is better to work in discrete-time [6].
There are attempts to calculate factors with continuous-time in literature [4], however calculations are prepared using exponential functions distributions. In engineering practice it very important to obtain accurate results without using strong simplifications.
The paper consists of discussions about the possibility and about the reason of carrying out these calculations, which is made by the application of simple models of the Markov and Semi-Markov processes, where there are attempts of use of continuous time in these calculations. Discussion is based on hypothetical exponential and non-exponential sojourn times' probabilities. Valuation of these methods is based on the comparison of availability and probabilities of transition values when using exponential and Weibull functions distributions. Previous experience presented in [11] gave a reason to estimate uncertainty of calculation method, when Weibull functions distributions and Semi-Markov solution and also continuous time are applied. This estimation is provided by example of very simple set, where there are two states of reliability. In sections 2 and 3 the hypothetical simple process is described by Markov and Semi-

Markov rules. Section 3 consists of description of prepared samples. Next there are going to be shown calculations.

2. Path's assumption

2.1. Markov approach

Let's make assumption that:

$P_1(t)$ - probability of sojourn time in up-state at the moment *t*;

$P_2(t)$ - probability of sojourn time in down-state at the moment *t*;

$\lambda(t)$ - intensity of failures;

$\mu(t)$ - intensity of repairs.

Below there is the matrix of intensities of transition

$$\Lambda = \begin{bmatrix} -\mu(t) & \mu(t) \\ \lambda(t) & -\lambda(t) \end{bmatrix} \tag{1}$$

The matrix can be described by a graph as shown in Figure 1. State 1 is assumed as up-state, state 2 is down-state.

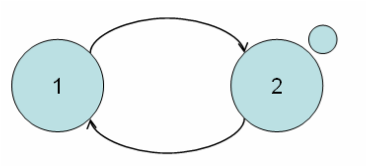

Figure. 1. Graph of state transition

The transition probability matrix is obtained by solving the Chapman-Kolmogorov equations:

$$P_1'(t) = P_2(t)\,\lambda(t) - \mu(t)\,P_1(t)$$

$$(2)$$

$$P_2'(t) = -P_2(t)\,\lambda(t) + \mu(t)\,P_1(t)$$

The stationary probabilities (the limiting probabilities) are given by well known formulas:

$$P_1 = \lim_{t\to\infty} P_1(t) = \frac{\lambda}{\mu+\lambda}$$

$$(4)$$

$$P_2 = \lim_{t\to\infty} P_2(t) = \frac{\mu}{\mu+\lambda}$$

2.2. Semi-Markov approach

There are three methods to define Semi – Markov processes [3], [4]:

1. by pair (***p***, **Q**(*t*)),

where: ***p*** – vector of initial distribution, **Q**(*t*) – matrix of distribution functions of transition times between states;

2. by triplets (***p***, ***P***, **F**(*t*)),

where: ***p*** – vector of initial distribution, ***P*** – matrix of transition probabilities, **F**(*t*) – matrix of distribution functions of sojourn times in state *i-th*, when *j-th* state is next;

3. by triplets (***p***, ***e***(*t*), **G**(*t*)), where: ***p*** – vector of initial distribution, ***e***(*t*) – matrix of probabilities of transition between *i-th* and *j-th* states, when sojourn time in state *i-th* is *x*, **G**(*t*) – matrix of unconditional sojourn times distribution functions.

The Markov process model that was included in this paper, in particular example of Semi-Markov is defined by (***p***, ***P***, **F**(*t*)).

Transition probabilities are one of the most important characteristics of Semi – Markov processes, which are defined as conditional probabilities

$$P_{ij}(t) = P\{X(t) = j \mid X(0) = i\},\ i, j \in S \tag{5}$$

These probabilities obey Feller's equations

$$P_{ij}(t) = \delta_{ij}[1 - G_i(t)] + \sum_{k\in S}\int_0^t P_{kj}(t-x)\,dQ_{ik}(x), \tag{6}$$

Solution of that set of equations can be found by applying the Laplace – Stieltjes transformation. After that transformation the set takes form

$$\tilde{p}_{ij}(s) = \delta_{ij}[1 - \tilde{g}_i(s)] + \sum_{k\in S}\tilde{q}_{ik}(s)\tilde{p}_{kj}(s),\quad i, j \in S, \tag{7}$$

In matrix form

$$\tilde{\mathbf{p}}(s) = [I - \tilde{\mathbf{g}}(s)] + \tilde{\mathbf{q}}(s)\tilde{\mathbf{p}}(s), \tag{8}$$

Hence

$$\tilde{\mathbf{p}}(s) = [I - \tilde{\mathbf{q}}(s)]^{-1}[I - \tilde{\mathbf{g}}(s)]\ . \tag{9}$$

3. Conditions determination for particular example

Assumed system, presented on figure 1, consist of two states. Object can stay in reliability states from the set S (0,1), where:

0 – unserviceability state,

1 – serviceability state.

First state is described by random variable ζ_{p1}. The distribution function of random variable is

$$F_{\zeta p1}(t) = P\{\zeta_{p1} \le t\},\ t \ge 0\,.$$

Normal activities can be interrupted by failures. If there is known time, when the system is broken down, and that time is given by χ_p, then the distribution function of state "repair" is

$$F_{\chi_p}(t) = P\{\chi_p \le t\},\ t \ge 0 .$$

The process can be described by Semi – Markov process $\{X(t) : t \ge 0\}$ with the finite set of states S_p = {1, 2}. The kernel of the process is described by matrix

$$\boldsymbol{Q_p}(t)= \begin{bmatrix} 0 & Q_{01} \\ Q_{10} & 0 \end{bmatrix}, \tag{10}$$

Transition from 1-st state to 2-nd can be described By

$$Q_{p01}(t) = p_{01} F_{\zeta_{p1}}(t),$$

Transition from 2-th state to 1-st:

$$Q_{p01}(t) = P(\chi_p < t) = F_{\chi_p}(t) .$$

The vector $\boldsymbol{p} = [p_1, p_2]$ is initial distribution of the process, in particular example $\boldsymbol{p} = [1,0]$.

4. Data and assumptions for calculations

For the purposes of the particular example, there was prepared two states set-up, which was mathematically described in section 2.1. It was described by Markov process and Semi-Markov process. Prepared data includes information about sojourn times, during 100 points of time. Initial value of sojourn time of serviceability is equal to 10, and after 100 observations this value decreases to value of 9.81. Each of following number is lower about 0.01. This set can describe the simple technical object, where the normal maintenance started before the earlier one, and last observation didn't finish one. Sojourn time of unserviceability is constant and is also equal to 1.

The data allow to asses main parameters characterizing sample according to exponential and Weibull functions distributions.

In practice simplifications based on assuming exponential distribution often is like routine. Consequently values of availability or transient probabilities are calculated basing on Markov process, with usage of mean values of sojourn state times or states transitions. This method is described in section 4.1.

Authors conducted calculation by assuming that the size of the sample and its character allow carrying out calculations basing on the mean values of sojourn time.

Basically the way of calculation is identical to the case described in point 4.1, but the prepared random sample was divided into 10 sections. For each of them, there were calculated parameters, that are important for authors.

In section 4.3 there is assumption, that transition from state 1 to state 2 is be described by Weibull function distribution, reverse transitions is exponential. In section 4.1 and 4.2 calculations are carried out with Markov procedures, in section 4.3 Semi-Markov procedures is applied. Necessary factors to make calculations are presented in *Table 1*, a parameter is close to value of 1 (close to exponential parameters).

Table 1. Distribution parameters for different distribution function

state 1	state 2
Parameter of exponential distribution (variant 1)	
λ=0.105	μ=1
Weibull and exponential distributions (variant 3)	
α=0.105	μ=1
β=1.1	

4.1 Markov calculation with one mean value

At first calculation has been done with assumption, that transient probabilities are exponential. Value of parameters is obtained from sample of 100. Mean time of sojourn time of state 1 is 9.52, intensity of transition between state 1 and 2 is 0.105, in reverse direction is constant and equal to 1. The distribution function of sojourn times and their Laplace – Stieltjes transformation take form:

$$F_{w1}(t) = 1 - e^{-0.105t}\ ,\ f_{w1}*(t) = \frac{0.105}{s + 0.105},$$

$$F_{w2}(t) = 1 - e^{-t},\ f_{w2}*(t) = \frac{1}{s+1}.$$

Then, kernel of the process is given by matrix

$$\boldsymbol{Q_p}(t) = \begin{bmatrix} 0 & 1 - e^{-0.105t} \\ 1 - e^{-t} & 0 \end{bmatrix}. \tag{8}$$

Matrices $\tilde{\mathbf{q}}(s)$ and $\tilde{\mathbf{g}}(s)$ have been determined according to equations (7) – (9). In considered example we obtain

$$\tilde{\mathbf{q}}(s) = \begin{bmatrix} 1 & \frac{0.105}{s+0.105} \\ \frac{1}{s+1} & 1 \end{bmatrix}$$

However, taking into account recent experience calculations will be carried out with Markov model with time going to infinity [9], [11]. In this case intensities of transition given for particular example are presented in *Table 2.*

Values P01 and P10 very quickly take stable values. In case of P10 it's after t=8. In particular example when sojourn time of state 1 decreases transient probability of transition to state 2 increases also.

Table 2. Intensities of transition in intervals

t	P00	P01	P10	P11
1	0.901	0.099	0.631	0.369
2	0.811	0.189	0.864	0.136
3	0.730	0.270	0.950	0.050
4	0.658	0.342	0.981	0.019
5	0.592	0.408	0.993	0.007
6	0.534	0.466	0.997	0.003
7	0.480	0.520	0.999	0.001
8	0.433	0.567	1	0
9	0.390	0.610	1	0
10	0.351	0.649	1	0
20	0.123	0.877	1	0
30	0.043	0.957	1	0
40	0.015	0.985	1	0
50	0.005	0.995	1	0

4.2. Example with exponential distributions and constant intensities in intervals

Taking into account prepared data authors assumed that value of intensity of transition will be calculated for each 10 samples. Having 100 observations it gives 10 intervals with variable mean values sojourn time of state 1st. In this case calculation procedures are the same like in previous variant. Introducing variable mean values prescribed to intervals, uncertainty of evaluation is expected to be smaller. Values of intensities of transition in intervals are presented in *Table 3*.

According to Markov calculation rules it is possible to obtain results presented in table 4. Parameter P0 can be treating as value of availability. There is also value of availability obtain in recent calculations. Parameter assigned by "*" represent availability results for previous example.

Table. 3. Results of states probabilities

t	λ	μ	P0	P1	P*
0	0.1000	1	0.9091	0.0909	0.9050
10	0.1005		0.9087	0.0913	
20	0.1015		0.9079	0.0921	
30	0.1025		0.9070	0.0930	
40	0.1035		0.9062	0.0938	
50	0.1046		0.9053	0.0947	
60	0.1056		0.9045	0.0955	
70	0.1067		0.9036	0.0964	
80	0.1077		0.9027	0.0973	
90	0.1088		0.9019	0.0981	
100	0.1099		0.9010	0.0990	

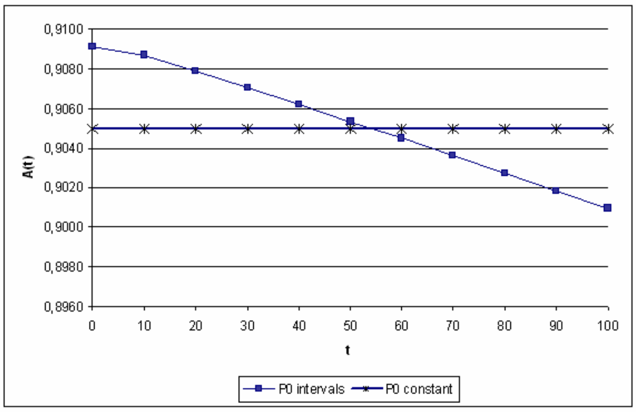

Figure 2. Comparison of values of availability obtained by Markov methods

On *Figure 2* representing results of comparison of two simple methods. It can be seen, that lowering number of intensities of transition causes degreasing value of transient probabilities and availability as time increasing. Calculating values of availability by first method can be treating as first estimation. Second way of calculating gives more detailed information.

Compared values of transient probabilities P01 , that were obtained by the application of two methods of calculating, did not result in serious differences. Sensitivity analysis did not give the clear answer, where question is which method gives certain values.

4.3. Example with Weibull distributions and Semi-Markov calculations

In third variant of calculation there was decided, that the sojourn time of state 1 is given by Weibull function distribution, and for state 2 it is exponential. According to table 2, collected data can be described by Weibull distributions. For particular calculations sojourn times distribution functions take form:

$$F_{b1}(x) = 1 - e^{-0.105t^{1.1}},$$

$$F_{b2}(x) = 1 - e^{-t}.$$

Derivative of Weibull distribution function (i.e. density function) is presented by

$$F'(t) = \lambda\alpha \cdot e^{-\lambda t^{\alpha}} \cdot t^{\alpha-1} \tag{16}$$

Laplace – Stieltjes transformations of Weibull distribution function can be obtained by using formula

$$\begin{aligned} f^*(t) &= \int_0^{\infty} e^{-st} \cdot F'(t)\,dt = \int_0^{\infty} e^{-st}(1-e^{-\lambda t^{\alpha}})'\,dt = \\ &= s\int_0^{\infty} e^{-st}(1-e^{-\lambda t^{\alpha}})dt - (1-e^{-\lambda \cdot 0^{\alpha}}) \end{aligned} \tag{17}$$

Hence

$$\begin{aligned} f^*(t) &= s\int_0^{\infty} e^{-st}(1-e^{-\lambda t^{\alpha}})dt = \\ &= s\int_0^{\infty} e^{-st}dt - s\int_0^{\infty} e^{-st} \cdot e^{-\lambda t^{\alpha}}dt = 1 - s\int_0^{\infty} e^{-st} \cdot e^{-\lambda t^{\alpha}}dt \end{aligned} \tag{18}$$

Using Maclaurin series for element "$e^{-\lambda t^{\alpha}}$", we obtain Laplace – Stieltjes transformation of the Weibull distribution function

$$\begin{aligned} f^*(t) &= \lambda\frac{\alpha \cdot \Gamma(\alpha)}{s^{\alpha}} - \frac{\lambda^2}{2!}\frac{2\alpha \cdot \Gamma(2\alpha)}{s^{2\alpha}} + \frac{\lambda^3}{3!}\frac{3\alpha \cdot \Gamma(3\alpha)}{s^{3\alpha}} - \ldots = \\ &= \sum_{n=1}^{\infty}\frac{\lambda^n}{n!}\frac{n\alpha \cdot \Gamma(n\alpha)}{S^{n\alpha}} \end{aligned} \tag{19}$$

For considered example, Weibull distribution Laplace – Stieltjes transformations take form, respectively

$$f_{b1}^*(t) = \frac{1}{s+1}$$

$$f_{b2}^*(t) = \frac{0.1083}{s^{1,1}} - \frac{0.1254}{s^{2,2}} + \frac{0.0016}{s^{3,3}} - \frac{0.0002}{s^{4,4}} + \ldots$$

Matrices $\tilde{\mathbf{q}}(s)$ and $\tilde{\mathbf{g}}(s)$ have been determined according to equations (5) – (7). In considered example obtain

$$\mathbf{Q}_p(t) = \begin{bmatrix} 0 & 1-e^{-0.105t^{1.1}} \\ 1-e^{-t} & 0 \end{bmatrix},$$

$$\tilde{\mathbf{q}}(s) = \begin{bmatrix} 1 & \frac{0,1083}{s^{1,1}} - \frac{0,1254}{s^{2,2}} \\ \frac{1}{s+1} & 1 \end{bmatrix},$$

$$\tilde{\mathbf{g}}(s) = \begin{bmatrix} \frac{0,1083}{s^{1,1}} - \frac{0,1254}{s^{2,2}} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix}.$$

State probability can be obtained form formula:

$$P_0 = 0.9066203925 - 0.8132407848 \cdot e^{-1,181495882t} - 0.09337960750 \cdot e^{-0,222144t}$$

Figure 3. presents graphs of state probabilities.

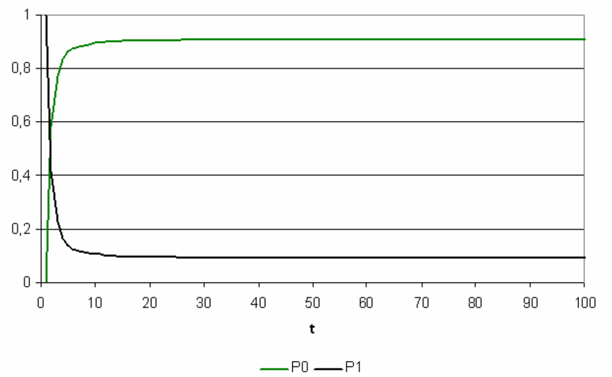

Figure 3. State probabilities graph with exponential and Weiubull functions.

Both functions distributions are non monotonic. Going to infinity they obtain constant values of P_1, P_2, calculated basing on Laplace transformation

$$P_i = \lim_{s\to 0} s\tilde{P}_i(s), \quad i = 1,2,$$

or calculated on the basis of ergodic theory for Semi-Markov processes. In particular example:

$$P_1 = 0.9066, \qquad P_2 = 0.0934$$

5. Conclusion

Semi - Markov processes allow for estimating crucial indices like availability or transition probabilities for systems, where the distributions functions are specified.

This article does not contain a straightforward answer, if the way of calculation and obtained results are correct. Parameters of these sets were matched in a way, to make it impossible to compare values of probabilities of transfer between states. On purpose there were not introduced any other states, because it could make the image disturbed and what is more it could lead to the illegibility of obtained result.

Obtained results show that the method of cutting off elements of Maclaurin's series is providing good quality results in the second attempt.

In case of Semi-Markov processes, usage of distribution function that is different from the exponential one, makes further calculations very complicated. However it is possible to obtain some results. Because of difficulties in calculation, profits from usage of Semi-Markov processes are limited. However simple models can be computed. The main goal of future work is the continuation of research which are connected with the applicability of Semi-Markov computation, where the distribution differs from exponential function. The improvement of analytical results gives the chance of preparation an accurate software simulator in the future, that simplifies calculation and decreases the level of uncertainties.

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