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Lifetime distributions with wave-like bathtub hazard

Keywords

lifetime distribution, hazard, bathtub hazard, likelihood function, maximum likelihood estimation

Abstract

In this paper, we argue the necessity of dealing with lifetime distributions with wave-like bathtub hazard function. Four classes of wave-like bathtub hazards are investigated. For preparing maximum likelihood estimation of the hazard parameters, the first-order and second-order partial derivatives are derived.

1. Introduction

Wave-like distributions have been applied to many engineering fields, e.g., Davidson and Frank [4], Idriss and Boulanger [10], King and Tucker [11], Lermo and Chávez-García [12].

Empirical evidences have shown that a repair/maintenance improves the system in certain degree [5]. Therefore the hazard function of a repairable system may reveal a wave-like pattern. In reliability engineering it got used to monotonic hazards [13]. Guo et al. [8] shows the evidence that maximum likelihood modeling of Kiln system functioning/failure(PM) data supports the wave-like distribution pattern believe.

Bathtub hazard function has obtained more and more attention in accelerate life testing, repairable system modeling, Love and Guo [14]. *Table 1* lists a few of bathtub hazard families Since the first proposed by Smith and Bain [15] in 1975. Logically, there are two forces acting on a repairable system, wearing out/damaging and recovering simultaneously. The former force causes system increasing in hazard, while the later force causes the system decreasing in hazard. Therefore the balance between the two forces let the system hazard possess a pattern of decreasing at the beginning, steady in the middle, and increasing at the end, i.e., bathtub pattern. Nevertheless the bathtub curve should not be expected smoothed locally. In other words, the hazard curve of a repairable should be wave-like bathtub shaped.

Table 1. Existing bathtub hazards

No.	$h(t)$	Authors
1	$\dfrac{\beta}{t+\gamma}+\delta t$	[9]
2	$\dfrac{\beta}{\alpha}\left(\dfrac{t}{\alpha}\right)^{\beta-1}\exp\left(\left(\left(\dfrac{t}{\alpha}\right)^{\beta}\right)\right)$	[15]
3	$\alpha+\beta t+\gamma t^2$	[1]
4	$\exp\left(\alpha+\beta t+\gamma t^2\right)$	[1]
5	$\dfrac{\beta}{t+\gamma}+\theta t^{\theta-1}$	[7]
6	$\theta+(1-\theta)(t+\theta)^{\theta-1}+\theta t^{\theta}$	[7]
7	$\theta+(1-\theta)(\pi t+\theta)^{\theta-1}+\theta(\pi t)^{\theta}$	[7]

Estimating the parameters of any particular statistical model is most often carried out via maximum likelihood estimation (MLE). Searching for the MLE is a basic and well understood procedure in reliability modeling. Any candidate of MLE's in general, is a local solution. Mathematically, any local solution will be an MLE, which maintains the elementary property of the MLE. Whether an MLE is a feasible solution is an engineering judgment. Therefore it is logical and preferred to search globally for candidate solutions to identify the best solution from an engineering (maintenance) perspective. The most popular global optimization algorithm is the genetic algorithm (GA). Recently, Cui et al. [2], [3] developed a new global searching scheme, called λ-algorithm, which is a simpler than that of GA with an equivalent searching efficiency.

Therefore, the structure of the remaining of the paper is stated as following: Section 2 to Section 5 will serve the developments of Type I to Type IV wave-like bathtub class respectively. Section 6 concludes this paper.

2. Type I wave-like bathtub

Type I wave-like bathtub hazard is proposed by Guo et al. [6] recently.

Definition 2.1. Denote the lifetime of a system by X taking values from $[0,+\infty)$. The density function is

$$f(x)=\left(\gamma+\frac{\sin^2(\alpha x)}{x^2\int_x^{+\infty}\frac{\sin^2\alpha y}{y^2}dy}\right)\exp\left(-\int_0^x\left(\gamma+\frac{\sin^2(\alpha s)}{s^2\int_s^{+\infty}\frac{\sin^2\alpha y}{y^2}dy}\right)ds\right) \quad (1)$$

Then X is called a lifetime with type I wave-like hazard function:

$$h(x)=\gamma+\sin^2(\alpha x)\Big/\left(x^2\int_x^{+\infty}\frac{\sin^2\alpha y}{y^2}dy\right) \quad (2)$$

Figure 1 shows that the Type I hazard demonstrates a wave-like bathtub pattern.

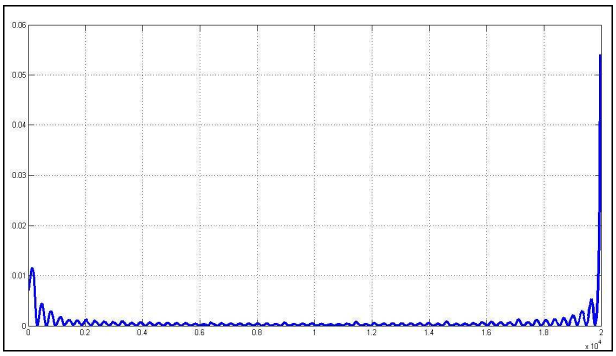

Figure 1. Hazard plot $h(t)$, $\alpha=0.01$, $\gamma=0$

Theorem 2.2. Let

$$\begin{aligned}A_{x_i}&=\gamma+\sin^2(\alpha x_i)\Big/\left(x_i^2\int_{x_i}^{+\infty}\frac{\sin^2(\alpha y)}{y^2}dy\right)\\ B_s&=\sin^2(\alpha s)\Big/\left(s^2\int_s^{+\infty}\frac{\sin^2(\alpha y)}{y^2}dy\right)\end{aligned} \quad (3)$$

Then the log-likelihood of Type I wave-like hazard can be expressed by

$$l(\alpha,\gamma\,|\,K)=\sum_{i=1}^N(1-\vartheta_i)A_{x_i}+\sum_{i=1}^N\int_0^{x_i}B_s ds \quad (4)$$

where $K=\{(x_i,\vartheta_i),i=1,2,\cdots,N\}$ is a sample with failure-censoring indicators,

$$\vartheta_i=\begin{cases}0 & x_i \text{ is a natural failure}\\ 1 & x_i \text{ is a censored event}\end{cases} \quad (5)$$

Furthermore, the first-order partial derivatives with respect to the two parameters are

$$\begin{aligned}\frac{\partial l}{\partial\alpha}&=\sum_{i=1}^N(1-\vartheta_i)\exp\left(-A_{x_i}\right)C_{x_i}-\sum_{i=1}^N\int_0^{x_i}C_s ds\\ \frac{\partial l}{\partial\gamma}&=\sum_{i=1}^N(1-\vartheta_i)\exp\left(-A_{x_i}\right)-\sum_{i=1}^N x_i\end{aligned} \quad (6)$$

where

$$C_u(\alpha)=\frac{\sin(2\alpha u)}{u\int_u^{+\infty}\left(\sin^2(\alpha s)/s^2\right)ds}-\frac{\sin^2(\alpha u)\int_u^{+\infty}\left(\sin(2\alpha s)/s\right)ds}{u^2\left(\int_u^{+\infty}\left(\sin^2(\alpha s)/s^2\right)ds\right)^2} \quad (7)$$

Finally, the second-order partial derivatives are

$$\begin{aligned}\frac{\partial^2 l}{\partial\alpha^2}&=\sum_{i=1}^N(1-\vartheta_i)\left[-\exp\left(-2A_{x_i}\right)C_{x_i}^2+\exp\left(-A_{x_i}\right)CA_{x_i}\right]\\&\quad-\sum_{i=1}^N\int_0^{x_i}CA_s ds\\ \frac{\partial^2 l}{\partial\alpha\partial\gamma}&=-\sum_{i=1}^N(1-\vartheta_i)\exp\left(-2A_{x_i}\right)C_{x_i}\\ \frac{\partial^2 l}{\partial\gamma^2}&=-\sum_{i=1}^N(1-\vartheta_i)\exp\left(-2A_{x_i}\right)\end{aligned} \quad (8)$$

where

$$CA_u(\alpha)=\frac{d}{d\alpha}C_u(\alpha)=CA_u^{(1)}(\alpha)+CA_u^{(2)}(\alpha) \quad (9)$$

with

$$CA_u^{(1)}(\alpha)=\frac{2\cos(2\alpha u)}{\int_u^{+\infty}\frac{\sin^2(\alpha y)}{y^2}dy}-\frac{\sin(2\alpha u)\int_u^{+\infty}\frac{\sin(2\alpha y)}{y}dy}{\left(\int_u^{+\infty}\frac{\sin^2(\alpha y)}{y^2}dy\right)^2} \tag{10}$$

and

$$CA_u^{(2)}(\alpha)=\frac{u\sin(2\alpha u)\int_u^{+\infty}\frac{\sin(2\alpha y)}{y}dy}{u^2\left(\int_u^{+\infty}\frac{\sin^2(\alpha y)}{y^2}dy\right)^2}+\frac{2\sin^2(\alpha u)\int_u^{+\infty}\cos(2\alpha y)dy}{u^2\left(\int_u^{+\infty}\frac{\sin^2(\alpha y)}{y^2}dy\right)^2}-\frac{2\sin^2(\alpha u)\left(\int_u^{+\infty}\frac{\sin(2\alpha y)}{y}dy\right)^2}{u^2\left(\int_u^{+\infty}\frac{\sin^2(\alpha y)}{y^2}dy\right)^3} \tag{11}$$

3. Type II wave-like bathtub

Type II wave-like bathtub hazard is also proposed by Guo et al. [6] recently.

Definition 3.1. Denote the lifetime of a system by X taking values from $[0,+\infty)$. The density function is

$$f(x)=\left(\gamma+\frac{\sin(\alpha x)}{x\int_x^{+\infty}\frac{\sin(\alpha y)}{y}dy}\right)\exp\left(-\int_0^x\left(\gamma+\frac{\sin(\alpha s)}{s\int_s^{+\infty}\frac{\sin(\alpha y)}{y}dy}\right)ds\right) \tag{12}$$

Then X is called a lifetime with type II wave-like hazard function

$$h(x)=\gamma+\frac{\sin(\alpha x)}{x\int_x^{+\infty}\frac{\sin(\alpha y)}{y}dy},\alpha,\gamma>0 \tag{13}$$

Figure 2 shows that the Type II hazard demonstrates a wave-like bathtub pattern.

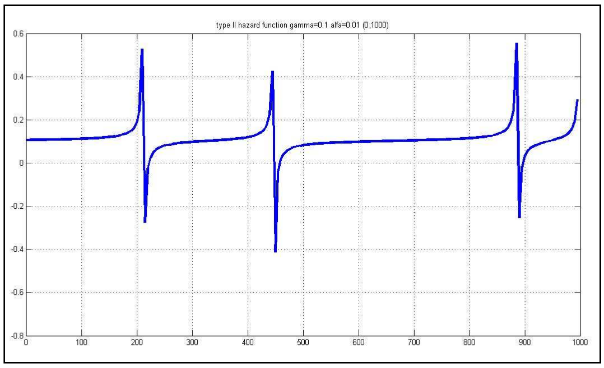

Figure 2. Hazard plot $h(t)$, $\alpha=0.01$, $\gamma=0.1$

Theorem 3.2. Let

$$M_u=\gamma+\sin(\alpha u)\Big/\left(u\int_u^{+\infty}\frac{\sin(\alpha y)}{y}dy\right)$$
$$N_s=\sin(\alpha u)\Big/\left(u\int_u^{+\infty}\frac{\sin(\alpha y)}{y}dy\right) \tag{14}$$

Then the log-likelihood of Type II wave-like hazard can be expressed by

$$l(\alpha,\gamma\,|\,K)=\sum_{i=1}^{n}(1-\vartheta_i)M_{x_i}+\sum_{i=1}^{n}\int_0^{x_i}N_s ds \tag{15}$$

where $K=\{(x_i,\vartheta_i),i=1,2,\cdots,N\}$ is a sample with failure-censoring indicators

$$\vartheta_i=\begin{cases}0 & x_i \text{ is a natural failure}\\ 1 & x_i \text{ is a censored event}\end{cases} \tag{16}$$

Furthermore, the first-order partial derivatives with respect to the two parameters are indicators

$$\frac{\partial l}{\partial\alpha}=\sum_{i=1}^{N}(1-\vartheta_i)\exp(-M_{x_i})R_{x_i}-\sum_{i=1}^{N}\int_0^{x_i}R_s ds$$
$$\frac{\partial l}{\partial\gamma}=\sum_{i=1}^{N}(1-\vartheta_i)\exp(-M_{x_i})-\sum_{i=1}^{N}x_i \tag{17}$$

where

$$R_u(\alpha)=\frac{\cos(\alpha u)}{\int_u^{+\infty}\frac{\sin^2(\alpha y)}{y^2}dy}-\frac{\sin(\alpha u)\int_u^{+\infty}\cos(\alpha y)dy}{\left(\int_u^{+\infty}\frac{\sin(\alpha y)}{y}dy\right)^2} \tag{18}$$

Finally, the second-order partial derivatives are

$$\begin{aligned}\frac{\partial^2 l}{\partial\alpha^2} &= \sum_{i=1}^{N}(1-\vartheta_i)\left[-\exp\left(-2M_{x_i}\right)R_{x_i}^2+\exp\left(-M_{x_i}\right)RA_{x_i}\right]\\ &\quad-\sum_{i=1}^{N}\int_0^{x_i}RA_s ds\\ \frac{\partial^2 l}{\partial\alpha\partial\gamma} &= -\sum_{i=1}^{N}(1-\vartheta_i)\exp\left(-2M_{x_i}\right)R_{x_i}\\ \frac{\partial^2 l}{\partial\gamma^2} &= -\sum_{i=1}^{N}(1-\vartheta_i)\exp\left(-2M_{x_i}\right)\end{aligned} \tag{19}$$

where

$$RA_u(a) = \frac{d}{da}R_u(a) = RA_u^{(1)}(a) + RA_u^{(2)}(a) \tag{20}$$

with

$$\begin{aligned}RA_u^{(1)}(\alpha) &= -\frac{u\sin(\alpha u)}{\int_u^{+\infty}\frac{\sin(\alpha y)}{y}dy}\\ &\quad-\frac{\cos(\alpha u)\int_u^{+\infty}\cos(\alpha y)dy}{\left(\int_u^{+\infty}\frac{\sin(\alpha y)}{y}dy\right)^2}\end{aligned} \tag{21}$$

and

$$\begin{aligned}RA_u^{(2)}(\alpha) &= -\frac{u\cos(\alpha u)\int_u^{+\infty}\cos(\alpha y)dy}{u\left(\int_u^{+\infty}\frac{\sin^2(\alpha y)}{y^2}dy\right)^2}\\ &\quad+\frac{\cos(\alpha u)\int_u^{+\infty}y\sin(2\alpha y)dy}{u^2\left(\int_u^{+\infty}\frac{\sin^2(\alpha y)}{y^2}dy\right)^2}\\ &\quad-\frac{2\sin(\alpha u)\left(\int_u^{+\infty}\cos(\alpha y)dy\right)^2}{u\left(\int_u^{+\infty}\frac{\sin^2(\alpha y)}{y^2}dy\right)^3}\end{aligned} \tag{22}$$

4. Type III wave-like bathtub

Type III wave-like bathtub hazard is a convex mixture of power law and Cauchy density with cosine term to catch up wave-like pattern.

Definition 4.1. Denote the lifetime of a system by X taking values from $[0,+\infty)$. The density function is

$$\begin{aligned}f(x) &= \left(\beta x^\gamma+(1-\beta)\frac{\cos(\alpha x)}{\sqrt{2}\left(1+x^2\right)}\right)\times\\ &\quad\exp\left(-\int_0^x\left(\beta s^\gamma+(1-\beta)\frac{\cos(\alpha s)}{\sqrt{2}\left(1+s^2\right)}\right)ds\right)\end{aligned} \tag{23}$$

Then X is called a lifetime with type III wave-like hazard function are

$$\begin{aligned}h(x) &= \beta x^\gamma+(1-\beta)\frac{\cos(\alpha x)}{\sqrt{2}\left(1+x^2\right)}\\ &\alpha,\gamma>0,\ \beta\in(0,1)\end{aligned} \tag{24}$$

Figure 3 shows that the Type III hazard demonstrates a wave-like bathtub pattern.

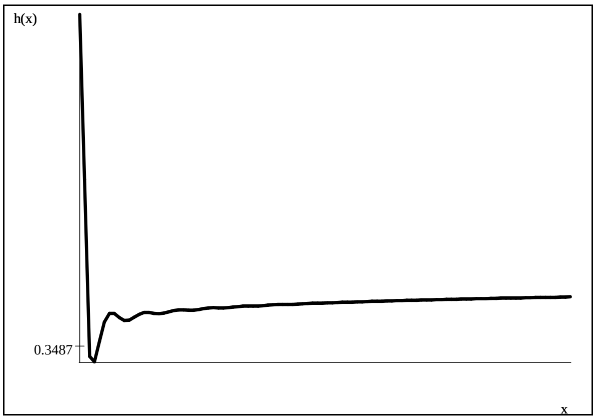

Figure 3. The plot of Type III wave-like bathtub hazard function ($(\alpha,\beta,\gamma)=(0.15\pi,0.35,0.08)$)

Remark 4.1. In equation (24), the term x^γ is monotone-increasing as long as $\gamma>0$, which catches up the wear-out/damage from the system operation, the term $1/(1+x^2)$ catches up the system repair/PM improvement to the system (average trend curve), and the term $\cos(\alpha x)$ may reveal the repair/damage related fluctuations about the average trend in hazard. If $\alpha=0$, the hazard becomes power law βx^γ.

Theorem 4.2. Let

$$\begin{aligned}P_u &= \ln\left(\beta u^\gamma+(1-\beta)\frac{\cos(\alpha u)}{\sqrt{2}\left(1+u^2\right)}\right)\\ Q_u &= \exp(P_u) = \beta u^\gamma+(1-\beta)\frac{\cos(\alpha u)}{\sqrt{2}\left(1+u^2\right)}\end{aligned} \tag{25}$$

Then the log-likelihood of Type III wave-like hazard can be expressed by

$$l(\alpha,\gamma\,|\,K) = \sum_{i=1}^{n}(1-\vartheta_i)P_{x_i}+\sum_{i=1}^{n}\int_0^{x_i}Q_s ds \tag{26}$$

where $K=\{(x_i,\vartheta_i), i=1,2,\cdots,N\}$ is a sample with failure-censoring indicators

$$\vartheta_i = \begin{cases}0 & x_i \text{ is a natural failure}\\ 1 & x_i \text{ is a censored event}\end{cases} \tag{27}$$

Furthermore, the first-order partial derivatives with respect to the two parameters are

$$\frac{\partial l}{\partial \alpha} = \sum_{i=1}^{n} (1-\vartheta_i) \exp(-P_{x_i}) QA_{x_i} - \sum_{i=1}^{n} \int_0^{x_i} QA_s \, ds$$
$$\frac{\partial l}{\partial \beta} = \sum_{i=1}^{n} (1-\vartheta_i) \exp(-P_{x_i}) QB_{x_i} - \sum_{i=1}^{n} \int_0^{x_i} QB_s \, ds \tag{28}$$
$$\frac{\partial l}{\partial \gamma} = \sum_{i=1}^{n} (1-\vartheta_i) \exp(-P_{x_i}) QC_{x_i} - \beta \sum_{i=1}^{n} x_i^{\gamma}$$

where

$$QA_u(\alpha,\beta) = -(1-\beta)\frac{u\sin(\alpha u)}{\sqrt{2}(1+u^2)}$$
$$QB_u(\alpha,\gamma) = u^{\gamma} - \frac{\cos(\alpha u)}{\sqrt{2}(1+u^2)} \tag{29}$$
$$QC_u(\alpha,\beta,\gamma) = \beta u^{\gamma}\ln(u)$$

Finally, the second-order partial derivatives are

$$\frac{\partial^2 l}{\partial \alpha^2} = \sum_{i=1}^{N} (1-\vartheta_i)\left[-\exp(-2P_{x_i}) QA_{x_i}^2 + \exp(-P_{x_i}) QAA_{x_i}\right]$$
$$-\sum_{i=1}^{N} \int_0^{x_i} QAA_s \, ds$$
$$\frac{\partial^2 l}{\partial \alpha \partial \beta} = \sum_{i=1}^{N} (1-\vartheta_i)\left[-\exp(-2P_{x_i}) QA_{x_i} QB_{x_i} + \exp(-P_{x_i}) QAB_{x_i}\right]$$
$$-\sum_{i=1}^{N} \int_0^{x_i} QAB_s \, ds$$
$$\frac{\partial^2 l}{\partial \alpha \partial \gamma} = \sum_{i=1}^{N} (1-\vartheta_i)\left[-\exp(-2P_{x_i}) QA_{x_i} QC_{x_i}\right]$$
$$\frac{\partial^2 l}{\partial \beta^2} = \sum_{i=1}^{N} (1-\vartheta_i)\left[-\exp(-2P_{x_i}) QB_{x_i}^2\right] \tag{30}$$
$$\frac{\partial^2 l}{\partial \beta \partial \gamma} = \sum_{i=1}^{N} (1-\vartheta_i)\left[-\exp(-2P_{x_i}) QB_{x_i} QC_{x_i} + \exp(-P_{x_i}) QBC_{x_i}\right]$$
$$-\sum_{i=1}^{N} x_i^{\gamma}$$
$$\frac{\partial^2 l}{\partial \gamma^2} = \sum_{i=1}^{N} (1-\vartheta_i)\left[-\exp(-2P_{x_i}) QC_{x_i}^2 + \exp(-P_{x_i}) QCC_{x_i}\right]$$
$$-\beta \sum_{i=1}^{N} \int_0^{x_i} s^{\gamma} \ln^2(s) \, ds$$

where

$$QAA_u(\alpha) = -(1-\beta)\frac{u^2\cos(\alpha u)}{\sqrt{2}(1+u^2)}$$
$$QAB_u(\alpha) = \frac{u\sin(\alpha u)}{\sqrt{2}(1+u^2)} \tag{31}$$
$$QBC_u(\gamma) = u^{\gamma}\ln(u)$$
$$QCC_u(\beta,\gamma) = \beta u^{\gamma}\ln^2(u)$$

with

$$\int_0^{x_i} s^{\gamma} \ln^2(s) ds$$
$$= x_i^{\gamma+1}\left(\frac{\ln^2(x_i)}{\gamma+1} + \frac{2\ln(x_i)}{(\gamma+1)^2} + \frac{2}{(\gamma+1)^3}\right) \tag{32}$$

5. Type IV wave-like bathtub

Type IV wave-like bathtub hazard is a convex mixture of power law and cosine-Cauchy density with cosine term to catch up wave-like pattern, which is an improved version from the two-parameter bathtub model [6].

Definition 5.1. Denote the lifetime of a system by X taking values from $[0,+\infty)$. The density function is

$$f(x) = \left(\gamma(1+x^{\gamma}) + (1-\gamma)\left(\frac{\cos(\alpha x)}{\sqrt{2}(1+x^2)} + (x+\gamma)^{\gamma-1}\right)\right) \times$$
$$\exp\left(-\int_0^x \left(\gamma(1+s^{\gamma}) + (1-\gamma)\left(\frac{\cos(\alpha s)}{\sqrt{2}(1+s^2)} + (s+\gamma)^{\gamma-1}\right)\right) ds\right) \tag{33}$$

Then X is called a lifetime with type IV wave-like hazard function

$$h(x) = \gamma(1+x^{\gamma}) + (1-\gamma)\left(\frac{\cos(\alpha x)}{\sqrt{2}(1+x^2)} + (x+\gamma)^{\gamma-1}\right) \tag{34}$$
$$\alpha, \ \gamma \in (0,1)$$

Remark 5.2. Intuitively, Type IV is a two-parameter wave-like bathtub family. Term x^{γ} is an increasing function as long as $\gamma > 0$; while term $(x+\gamma)^{\gamma-1}$ is a decreasing function as long as $\gamma \in (0,1)$. The third term $\cos(\alpha x)/\left(\sqrt{2}(1+x^2)\right)$ is a decreasing function which fluctuates around the decreasing curve $\cos(\alpha x)/\left(\sqrt{2}(1+x^2)\right)$ as x increases.

Figure 4 shows that the Type III hazard demonstrates a wave-like bathtub pattern.

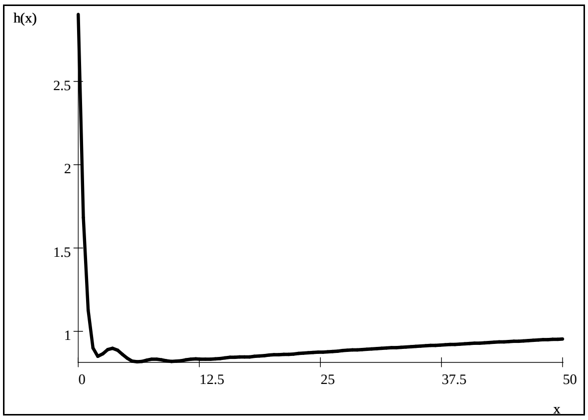

Figure 4. The plot of Type IV wave-like bathtub hazard function ($(\alpha,\gamma) = (0.5\pi, 0.25)$)

Theorem 5.3. Let

$$X_u = \ln\left(\gamma(1+x^{\gamma}) + (1-\gamma)\left(\frac{\cos(\alpha x)}{\sqrt{2}(1+x^2)} + (x+\gamma)^{\gamma-1}\right)\right)$$
$$Y_u = \exp(X_u) = \gamma(1+x^{\gamma}) + (1-\gamma)\left(\frac{\cos(\alpha x)}{\sqrt{2}(1+x^2)} + (x+\gamma)^{\gamma-1}\right) \tag{35}$$

Then the log-likelihood of Type IV wave-like hazard can be expressed by

$$l(\alpha,\gamma \mid K)=\sum_{i=1}^{n}\left(1-\vartheta_i\right)X_{x_i}+\sum_{i=1}^{n}\int_0^{x_i}Y_s ds \tag{36}$$

where $K=\{(x_i,\vartheta_i), i=1,2,\cdots,N\}$ is a sample with failure-censoring indicators

$$\vartheta_i=\begin{cases}0 & x_i \text{ is a natural failure}\\ 1 & x_i \text{ is a censored event}\end{cases} \tag{37}$$

Furthermore, the first-order partial derivatives with respect to the two parameters are

$$\begin{aligned}\frac{\partial l}{\partial\alpha}&=\sum_{i=1}^{n}\left(1-\vartheta_i\right)\exp\left(-P_{x_i}\right)YA_{x_i}-\sum_{i=1}^{n}\int_0^{x_i}YA_s ds\\ \frac{\partial l}{\partial\gamma}&=\sum_{i=1}^{n}\left(1-\vartheta_i\right)\exp\left(-P_{x_i}\right)YC_{x_i}-\sum_{i=1}^{n}\int_0^{x_i}YC_s ds\end{aligned} \tag{38}$$

where

$$\begin{aligned}&YA_u(\alpha,\gamma)=-(1-\gamma)\frac{u\sin(\alpha u)}{\sqrt{2}\left(1+u^2\right)}\\ &YC_u(\alpha,\gamma)=1+u^{\gamma}+u^{\gamma}\ln(u)\\ &-\left(\frac{\cos(\alpha u)}{\sqrt{2}\left(1+u^2\right)}+(u+\gamma)^{\gamma-1}\right)+(1-\gamma)\times\\ &\left((u+\gamma)^{\gamma-1}\ln(u+\gamma)+(\gamma-1)(u+\gamma)^{\gamma-2}\right)\end{aligned} \tag{39}$$

Finally, the second-order partial derivatives are

$$\begin{aligned}\frac{\partial^2 l}{\partial\alpha^2}&=\sum_{i=1}^{N}(1-\vartheta_i)\left[-\exp\left(-2P_{x_i}\right)YA_{x_i}^2+\exp\left(-P_{x_i}\right)YAA_{x_i}\right]\\ &-\sum_{i=1}^{N}\int_0^{x_i}YAA_s ds\\ \frac{\partial^2 l}{\partial\alpha\partial\gamma}&=\sum_{i=1}^{N}(1-\vartheta_i)\left[-\exp\left(-2P_{x_i}\right)YA_{x_i}YC_{x_i}+\exp\left(-P_{x_i}\right)YAC_{x_i}\right]\\ &-\sum_{i=1}^{N}\int_0^{x_i}YAC_s ds\\ \frac{\partial^2 l}{\partial\gamma^2}&=\sum_{i=1}^{N}(1-\vartheta_i)\left[-\exp\left(-2P_{x_i}\right)YC_{x_i}^2+\exp\left(-P_{x_i}\right)YCC_{x_i}\right]\\ &-\sum_{i=1}^{N}\int_0^{x_i}YCC_s ds\end{aligned} \tag{40}$$

where

$$\begin{aligned}&YAA_u(\alpha,\gamma)=-(1-\gamma)\frac{u^2\cos(\alpha u)}{\sqrt{2}\left(1+u^2\right)}\\ &YAC_u(\alpha,\gamma)=\frac{u\sin(\alpha u)}{\sqrt{2}\left(1+u^2\right)}\\ &YCC_u(\gamma)=2u^{\gamma}\ln(u)+\gamma u^{\gamma}\ln^2(u)\\ &+(\gamma+u)^{\gamma-1}\ln(\gamma+u)-(\gamma-1)(\gamma+u)^{\gamma-2}\\ &+Z_u^{(1)}(\gamma)+Z_u^{(2)}(\gamma)+Z_u^{(3)}(\gamma)\end{aligned} \tag{41}$$

with

$$\begin{aligned}&Z_u^{(1)}(\gamma)=(1-\gamma)(2(\gamma+u)^{\gamma-2}+(\ln(\gamma+u))'\\ &((\ln(\gamma+u))(\gamma+u)^{\gamma-1}+(\gamma-1)(\gamma+u)^{\gamma-2}))\\ &Z_u^{(2)}(\gamma)=(1-\gamma)(\gamma-1)((\ln(\gamma+u))(\gamma+u)^{\gamma-2}\\ &\qquad+(\gamma-2)(\gamma+u)^{\gamma-3})\\ &Z_u^{(3)}(\gamma)=-(\gamma-1)(\gamma+u)^{\gamma-2}-(\ln(\gamma+u))(\gamma+u)^{\gamma-1}\end{aligned} \tag{42}$$

6. Conclusion

The idea for introducing wave-like lifetime distribution was already causing some concern from certain corner of the reliability researchers who committed their whole life for using smoothing types of lifetime distributions. As to the wave-like bathtub hazards, it is seems offensive for those traditional reliability researchers. To us the authors of this paper, it is merely a collision between an ideology of mathematical convenience and the existence facing the diversifying real world. As a matter of fact smoothing bathtub function appears in very rare circumstances, for example, the hydrogen turbine generation unit, where once the unit passed its early warming stage, its operational speed is constant. However, in public transportation, in the warfare related tanks, cannons and even the ships the load there are constantly altered and the hazards too.

In our paper, we offered the four classes of wave-like bathtub hazards and accordingly the MLE information. It is obvious that the integration problem will cost computation time heavily in searching MLE. To get an effective remedy will be our aim in the next research stage.

Acknowledgments

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