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Semi-Markov model of system damage process

Keywords

reliability, multi-state, semi-Markov, damages processes

Abstract

The reliability characteristics and parameters of multi-state systems modelled by the finite states regress characteristics and parameters of multi-state systems modelled by the finite states regress semi-Markov processes are investigated in the paper. Presented here models deal with un-repairable systems. The essential concepts of discrete states and continuous time semi-Markov process theory deliver. Mathematical apparatus for models constructions and analysis. Multi-state reliability functions and corresponding expectations, second moments and standard deviations are calculated for the presented systems.

1. Introduction

Some concepts of a semi-Markov process theory [2], [3], [4], [7], [9], [12] are applied to construct a reliability model of an object. Markov and semi-Markov processes for modelling multi-state systems are applied in many different reliability problems [1], [4], [5], [6], [7], [8], [10], [11]. We will consider systems with finite sets of the ordered reliability states $S = \{0, 1, \dots n\}$, where the state 0 is the worst while the state n is the best. We suppose that the probabilistic model of reliability evolution of the system is a stochastic process $\{X(t) :\ t \geqslant 0\}$, taking values in a state set $S = \{0, 1, \dots n\}$, with the right continuous trajectories and a flow graph of which is a coherent sug-graph of the graph shown in *Figure 1*. That kind of stochastic process is called a process of regress. Examples those kind of graphs are shown in *Figure 2* and *Figure 3*.

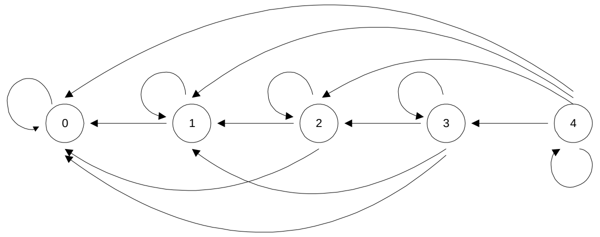

Figure 1. A general flow graph of a regress process

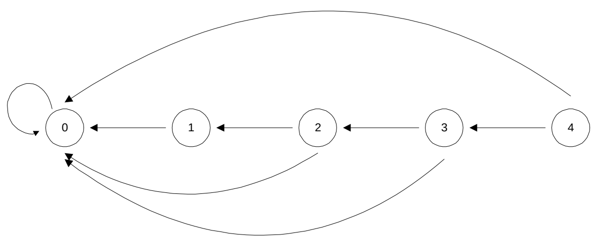

Figure 2. Example of a flow graph of a regress process

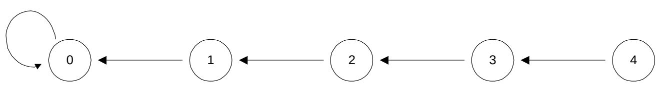

Figure 3. Example of a flow graph of a regress process

2. Essential concepts of a discrete states and continuous time semi-Markv processes theory

The semi-Markov processes were introduced independently and almost simultaneously by P. Levy, W.L. Smith, and L.Takacs in 1954-55. The essential developments of the semi-Markov processes theory were proposed by Koroluk & Turbin [7], [8], Limnios & Oprisan [9]. We will present only semi-Markov processes with a finite state space. Usually a semi-Markov process are constructed by the so called Markov Renewal Chain $\{\xi_n, \vartheta_n : n \in N_0\}$,

$\xi_n \in S,\ \vartheta_n \in [0,\infty)$, which is a special case of two-dimensional Markov sequence, such that the transition probabilities depend only on the discrete coordinate:

$$P(\xi_{n+1} = j, \vartheta_{n+1} \leqslant t \,|\, \xi_n = i, \vartheta_n = t_n\}) = \\ = P(\xi_{n+1} = j, \vartheta_{n+1} \leqslant t \,|\, \xi_n = i)$$

and

$$P(\xi_0 = i, \vartheta_0 = 0\} = P\{\xi_0 = i).$$

The matrix

$$Q(t) = [Q_{ij}(t) : \quad i,j \in S], \quad t \geqslant 0, \qquad (1)$$

where

$$Q_{ij}(t) = P(\xi_{n+1} = j, \vartheta_{n+1} \leqslant t \,|\, , \xi_n = i) \qquad (2)$$

is said to be the renewal kernel. Let

$$\tau_0 = 0, \quad \tau_n := \vartheta_1 + \ldots + \vartheta_n. \qquad (3)$$

The stochastic processes $\{\nu(t) : t \geqslant 0\}$, given by

$$\nu(t) = n \quad \text{for} \quad t \in [\tau_n, \tau_{n+1}), \quad n \in N_0. \qquad (4)$$

is called counting process.
The stochastic process $\{X(t) : t \geqslant 0\}$, defined by the formula

$$X(t) = \xi_n \quad \text{for} \quad t \in [\tau_n, \tau_{n+1}), \quad n \in N_0. \qquad (5)$$

is said to be the semi-Markov process given by the renewal kernel $Q(t)$.
From the above definition it follows that the semi-Markov processes keep constant values on the half-intervals. From the definition of the semi-Markov process it follows that the sequence $\{X(\tau_n) :\ n = 0, 1, \ldots\}$ is a homogeneous Markov chain with transition probabilities

$$p_{ij} = P(X(\tau_{n+1}) = j | X(\tau_n) = i) = \lim_{t\to\infty} Q_{ij}(t). \qquad (6)$$

The function

$$G_i(t) = P(\tau_{n+1} - \tau_n \leqslant t | X(\tau_n) = i) = \sum_{j \in S} Q_{ij}(t). \qquad (7)$$

is a cumulative probability distribution of a random variable T_i that is called a waiting time of the state i. The waiting time T_i is the time spent in state i when the successor state is unknown. The function

$$F_{ij}(t) = P(\tau_{n+1} - \tau_n \leqslant t | X(\tau_n) = i, X(\tau_{n+1}) = j) \qquad (8)$$

is a cumulative probability distribution of a random variable T_{ij} that is called a holding time of a state i, if the next state is j. From (6) we have

$$Q_{ij}(t) = p_{ij} F_{ij}(t). \qquad (9)$$

From (7) it follows that a semi-Markov process with a discrete state space can be defined by the transition matrix of the embedded Markov chain $P = [p_{ij} :\ i,j \in S]$ and a matrix of CDF of holding times: $F(t) = [F_{ij}(t) :\ i,j \in S]$. A semi-Markov process $\{X(t) :\ t \geqslant 0\}$ is said to be regular if the corresponding counting process $\{\nu(t) :\ t \geqslant 0\}$ has a finite number of jumps on a finite period with probability 1:

$$\bigwedge_{t \in \mathbb{R}_+} P(\nu(t) < \infty) = 1. \qquad (10)$$

Every semi-Markov process with a finite state space is regular [8].
Let

$$P_{iB}(t) = P(X(u) \in B,\ \forall u \in [0,\, t] \,|\, X(0) = i), \\ i \in B. \qquad (11)$$

denotes a probability that the whole time of $[0,\, t]$ the states of the process belong to a subset *B*, if an initial state is $i \in B$.
As a conclusion from theorem 3.9 [9] we obtain a theorem:
Functions $P_{iB}(t),\ \ i \in B \subset S,$ *satisfy system of integral equations*

$$P_{iB}(t) = 1 - G_i(t) + \sum_{j \in B} \int_0^t P_{jB}(t-x)\, dQ_{ij}(x), \\ i \in B. \qquad (12)$$

Using Laplace transformation we obtain system of linear equation

$$\tilde{P}_{iB}(s) = \frac{1}{s} - \tilde{G}_i(s) + \sum_{j \in B} \tilde{q}_{ij}(s) \tilde{P}_{jB}(s), \\ i \in B. \qquad (13)$$

where

$$\tilde{q}_{ij}(s) = \int_0^\infty e^{-st} dQ_{ij}(t).\ \tilde{G}_i(s) = \int_0^\infty e^{-st} G_i(t) dt,$$

$$\tilde{P}_{iB}(s) = \int_0^\infty e^{-st} P_{iB}(t)dt.$$

If B is a subset of the working states then the function

$$R_i(t) = P_{iB}(t), \quad i \in B \subset S \tag{14}$$

is the reliability function of a system with the initial state $i \in B$ at $t = 0$.
The conditional reliability functions satisfy the system of integral equation

$$R_i(t) = 1 - G_i(t) + \sum_{j \in B} \int_0^t R_j(t-x)\, dQ_{ij}(v), \quad i \in B \tag{15}$$

Using the Laplace transformation we obtain the system of linear equations

$$\tilde{R}_i(s) = \frac{1}{s} - \tilde{G}_i(s) + \sum_{j \in B} \tilde{q}_{ij}(s)\tilde{R}_j(s), \quad i \in B. \tag{16}$$

The inverse Laplace transforms of the function which are the solution of the above system equations are the conditional reliability functions

$$R_i(t) = P\{T > t \,|\, X(0) = i\}, \quad i \in B \tag{17}$$

where T is a random variable denoting a lifetime of the system. Applying formulas

$$\begin{aligned} E(T|X(0) = i) &= \lim_{s \to 0} \tilde{R}_i(s) \\ E(T^2|X(0) = i) &= -2 \lim_{s \to 0} [\tilde{R}'_i(s)] \end{aligned} \tag{18}$$

we obtain a conditional Mean Time to Failures and corresponding Second Moment

3. General semi-Markov model of the system damage

We suppose that the state of the system is described by the semi-Markov process which is defined by the renewal kernel

$$\mathbf{Q}(t) = \begin{bmatrix} Q_{00}(t) & 0 & 0 & \cdots & 0 \\ Q_{10}(t) & Q_{11}(t) & 0 & \cdots & 0 \\ Q_{20}(t) & Q_{21}(t) & Q_{22}(t) & \cdots & 0 \\ Q_{30}(t) & Q_{31}(t) & Q_{32}(t) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 \\ Q_{n0}(t) & Q_{n\,1}(t) & \cdots & \cdots & Q_{n\,n}(t) \end{bmatrix}. \tag{19}$$

A corresponding flow graph for $n = 4$ is shown in *Figure 1*. Let

$$T_{[l]} = \inf\{t : X(t) \in A_{[l]}\} \tag{20}$$

where

$$A_{[l]} = \{0, \ldots, l-1\} \quad \text{and} \quad A'_{[l]} = \{l, \ldots, n\}.$$

The function

$$\Phi_{i[l]}(t) = P(T_{[l]} \leqslant t | X_(0) = i), \quad i \in A'_{[l]} \tag{21}$$

represents the cumulative distribution function (CDF) of the first passage time from the state $i \in A'_{[l]}$ to the subset $A_{[l]}$ for $\{X(t) : t \geqslant 0\}$. If $X(0) = n$, then the random variable $T_{[l]}$ represents the lifetime of the one component system in the subset $A'_{[l]}$. A corresponding reliability function has a form

$$R_{n[l]}(t) = P(T_{[l]} > t | X(0) = n) = 1 - \Phi_{nA_{[l]}}(t). \tag{22}$$

On the other hand

$$\begin{aligned} &P(T_{[l]} > t | X(0) = n) = \\ &= P(X(u) \in A'_{[l]}, \forall u \in [0,t] \,|\, X(0) = n). \end{aligned} \tag{23}$$

In this case we have

$$\begin{aligned} &P(T_{[l]} > t | X(0) = n) = \\ &= P(X(t) \in A'_{[l]} \,|\, X(0) = n). \end{aligned} \tag{24}$$

Applying equations (16) we obtain the system of linear equations for the Laplace transform of reliability functions

$$\tilde{R}_{i\,[l]}(s) = \frac{1}{s} - \tilde{G}_i(s) + \sum_{j \in A'_{[l]}} \tilde{q}_{ij}(s)\tilde{R}_{j\,[l]}(s), \quad i \in A'_{[l]}. \tag{25}$$

where

$$\tilde{G}_i(s) = \int_0^\infty e^{-st} G_i(t)dt, \quad \tilde{R}_{i\,[l]}(s) = \int_0^\infty e^{-st} R_{i\,[l]}(t)dt$$

are the Laplace transforms of the functions

$G_i(t),\ R_{i\,[l]}(t),\ t \geqslant 0$. Passing to matrix form we get

$$\left(\mathbf{I} - \tilde{q}_{A'_{[l]}}(s)\right) \tilde{\mathbf{R}}_{A'_{[l]}}(s) = \tilde{\mathbf{G}}_{A'_{[l]}}(s) \tag{26}$$

where

$$\mathbf{I} = \left[\delta_{ij} :\ i, j \in A'_{[l]}\right]$$

is the unit matrix,

$$\tilde{\mathbf{q}}_{A'_{[l]}}(s) = \left[\tilde{q}_{ij}(s) :\ i, j \in A'_{[l]}\right], \tag{27}$$

$$\tilde{\mathbf{G}}_{A'_{[l]}}(s) = \frac{1}{s}\left[1 - \sum_{j \in S} \tilde{q}_{ij}(s) :\ i \in A'_{[l]}\right]^T,$$

$$\tilde{\mathbf{R}}_{A'_{[l]}}(s) = \left[R_{i\,[l]} :\ i \in A'_{[l]}\right]^T$$

A vector function

$$\tilde{\mathbf{R}}(s) = \left[\tilde{R}_{n\,[0]}(s), \tilde{R}_{n\,[1]}(s), \ldots, \tilde{R}_{n\,[n]}(s)\right] \tag{28}$$

is a Laplace transform of multi-state reliability function of the system.
Example 1

Let $S = \{0, 1, 2, 3\}$.

Hence

$$\begin{array}{ll} A_{[1]} = \{0\}, & A'_{[1]} = \{1, 2, 3\}, \\ A_{[2]} = \{0, 1\}, & A'_{[2]} = \{2, 3\}, \\ A_{[3]} = \{0, 1, 2\}, & A'_{[3]} = \{3\}. \end{array}$$

The matrices from equation (35) for l=1 take form

$$\mathbf{I} - \tilde{q}_{A'_{[1]}}(s) = \begin{bmatrix} 1 - \tilde{q}_{11}(s) & 0 & 0 \\ \tilde{q}_{21}(s) & 1 - \tilde{q}_{22}(s) & 0 \\ \tilde{q}_{31}(s) & \tilde{q}_{32}(s) & 1 - \tilde{q}_{33}(s) \end{bmatrix}, \tag{29}$$

$$\tilde{\mathbf{G}}_{A'_{[1]}}(s) = \frac{1}{s}\begin{bmatrix} 1 - \tilde{q}_{10}(s) - \tilde{q}_{11}(s) \\ 1 - \tilde{q}_{20}(s) - \tilde{q}_{21}(s) - \tilde{q}_{22}(s) \\ 1 - \tilde{q}_{30} - \tilde{q}_{31}(s) - \tilde{q}_{32}(s) - \tilde{q}_{33}(s) \end{bmatrix}$$

We are interested in an element $\tilde{R}_{3[1]}(s)$ of the solution

$$\tilde{\mathbf{R}}_{A'[l]}(s) = \begin{bmatrix} \tilde{R}_{1[1]}(s) \\ \tilde{R}_{2[1]}(s) \\ \tilde{R}_{3[1]}(s) \end{bmatrix}. \tag{30}$$

This Laplace transform is

$$\tilde{R}_{3[1]}(s) = \frac{\tilde{u}_3(s)}{s(1 - \tilde{q}_{11}(s))(1 - \tilde{q}_{22}(s))(1 - \tilde{q}_{33}(s))}, \tag{31}$$

where

$$\begin{aligned} &\tilde{u}_3(s) = 1 - \tilde{q}_{11}(s) - \tilde{q}_{22}(s) + \tilde{q}_{11}(s)\tilde{q}_{22}(s) - \tilde{q}_{30}(s) \\ &+\tilde{q}_{11}(s)\tilde{q}_{30(s)} + \tilde{q}_{22}(s)q_{30}(s) - \tilde{q}_{11}(s)\tilde{q}_{22}(s)\tilde{q}_{30}(s) \\ &-\tilde{q}_{10}(s)\tilde{q}_{31}(s) + \tilde{q}_{10}(s)\tilde{q}_{22}(s)\tilde{q}_{31}(s) - \tilde{q}_{20}(s)\tilde{q}_{32}(s) \\ &+\tilde{q}_{11}(s)\tilde{q}_{20}(s)\tilde{q}_{32}(s) - \tilde{q}_{10}(s)\tilde{q}_{21}(s)\tilde{q}_{32}(s) - \tilde{q}_{33}(s) \\ &+\tilde{q}_{11}(s)q_{33}(s) + \tilde{q}_{22}(s)\tilde{q}_{33}(s) - \tilde{q}_{11}(s)\tilde{q}_{22}(s)\tilde{q}_{33}(s) \end{aligned} \tag{32}$$

The matrices from equations (30) for *l*=2 take form

$$\mathbf{I} - \tilde{q}_{A'_{[2]}}(s) = \begin{bmatrix} 1 - \tilde{q}_{22}(s) & 0 \\ \tilde{q}_{32}(s) & 1 - \tilde{q}_{33}(s) \end{bmatrix},$$

$$\tilde{\mathbf{G}}_{A'_{[2]}}(s) = \frac{1}{s}\begin{bmatrix} 1 - \tilde{q}_{20}(s) - \tilde{q}_{21}(s) - \tilde{q}_{22}(s) \\ 1 - \tilde{q}_{30} - \tilde{q}_{31}(s) - \tilde{q}_{32}(s) - \tilde{q}_{33}(s) \end{bmatrix}$$

By solving (26) we get

$$\tilde{R}_{3[2]}(s) = \frac{\tilde{u}_2(s)}{s\,(1 - \tilde{q}_{22}(s))(1 - \tilde{q}_{33}(s))}, \tag{33}$$

where

$$\begin{aligned} &\tilde{u}_2(s) = 1 - \tilde{q}_{22}(s) - \tilde{q}_{30}(s) - \tilde{q}_{33}(s) - \tilde{q}_{31}(s) \\ &+\tilde{q}_{22}(s)\tilde{q}_{31}(s) - \tilde{q}_{20}(s)\tilde{q}_{32}(s) - \tilde{q}_{21}(s)\tilde{q}_{32}(s) \\ &+\tilde{q}_{22}(s)\tilde{q}_{33}(s)) + \tilde{q}_{22}(s)\tilde{q}_{30}(s) \end{aligned} \tag{34}$$

The matrices from (26) for *l*=3 take forms

$$\mathbf{I} - \tilde{q}_{A'_{[3]}}(s) = \begin{bmatrix} 1 - \tilde{q}_{33}(s) \end{bmatrix},$$

$$\tilde{\mathbf{G}}_{A'_{[3]}}(s) = \frac{1}{s}\begin{bmatrix} 1 - \tilde{q}_{30} - \tilde{q}_{31}(s) - \tilde{q}_{32}(s) - \tilde{q}_{33}(s) \end{bmatrix}$$

(1)

Now, a solution of (26) is

$$\tilde{R}_{3[3]}(s) = \frac{1 - \tilde{q}_{30} - \tilde{q}_{31}(s) - \tilde{q}_{32}(s) - \tilde{q}_{33}(s)}{s\,(1 - \tilde{q}_{33}(s))}, \quad (35)$$

For lots of cases the elements $Q_{ii}(t),\ i = 1, 2, \ldots, n$ are equal to 0. Let us suppose that

$$\mathbf{Q}(t) = \begin{bmatrix} Q_{00}(t) & 0 & 0 & 0 \\ Q_{10}(t) & 0 & 0 & 0 \\ Q_{20}(t) & Q_{21}(t) & 0 & 0 \\ Q_{30}(t) & Q_{31}(t) & Q_{32}(t) & 0 \end{bmatrix}. \quad (36)$$

From (39) – (44) we obtain

$$\tilde{R}_{3[1]}(s) = \frac{\tilde{u}_3(s)}{s}, \quad (37)$$

$$\tilde{u}_3(s) = 1 - \tilde{q}_{30}(s) - \tilde{q}_{10}(s)\tilde{q}_{31}(s) \\ -\tilde{q}_{20}(s)\tilde{q}_{32}(s) - \tilde{q}_{10}(s)\tilde{q}_{21}(s)\tilde{q}_{32}(s),$$

$$\tilde{R}_{3[2]}(s) = \frac{\tilde{u}_2(s)}{s}, \quad (38)$$

$$\tilde{u}_2(s) = 1 - \tilde{q}_{30}(s) - \tilde{q}_{31}(s) \\ -\tilde{q}_{20}(s)\tilde{q}_{32}(s) - \tilde{q}_{21}(s)\tilde{q}_{32}(s)$$

and

$$\tilde{R}_{3[3]}(s) = \frac{1 - \tilde{q}_{30} - \tilde{q}_{31}(s) - \tilde{q}_{32}(s)}{s}. \quad (39)$$

A Laplace transform of the multi-state reliability function of that system is

$$\tilde{\mathbf{R}}(s) = \left[\tilde{R}_{3\,[0]}(s), \tilde{R}_{3\,[1]}(s), \tilde{R}_{3\,[2]}(s), \tilde{R}_{3\,[3]}(s)\right].$$

4. Multi-state model of two kind of failures

We assume that the failures are caused of wear or by some random events. There are possible only the state changes from k to $k-1$ or from k to 0 with the positive probabilities (*Figure 2*). Time of change from a state k to $k-1,\ k = 1, \ldots, n$ because of wear is assumed to be a nonnegative random variable η_k with a PDF $f_k(x),\ x \geqslant 0$. Time to a total failure (state 0) for the system in the state k is a nonnegative random variable ζ_k exponentially distributed with a parameter λ_k. Under those assumptions the stochastic process $\{X(t):\ t \geqslant 0\}$, describing the reliability state changes of the system, is the semi-Markov process with a state space $S = \{0, 1, \ldots, n\}$ and a kernel

$$\mathbf{Q}(t) = \begin{bmatrix} Q_{00}(t) & 0 & 0 & \cdots & 0 \\ Q_{10}(t) & 0 & 0 & \cdots & 0 \\ Q_{20}(t) & Q_{21}(t) & 0 & \cdots & 0 \\ Q_{30}(t) & 0 & Q_{32}(t) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 \\ Q_{n0}(t) & 0 & \cdots & Q_{n\,n-1}(t) & 0 \end{bmatrix}.$$

where

$$Q_{k\,k-1}(t) = P(\eta_k \leqslant t, \zeta_k > \eta_k) = \int_0^t e^{-\lambda_k x} f_k(x)dx$$

$$Q_{k\,0}(t) = P(\zeta_k \leqslant t, \eta_k > \zeta_k) = \int_0^t \lambda_k e^{-\lambda_k x}[1 - F_k(x)]dx$$

for $k = 1, \ldots, n$.
To explain this model we assume that $n = 3$ and we suppose that the random variables $\eta_k,\ k = 1, 2, 3$ have the gamma distribution with parameters $\alpha_k = 1, 2, \ldots$ and $\beta_k > 0$ with PDF

$$f_k(x) = \frac{\beta_k^{\alpha_k}\, x^{\alpha_k - 1} e^{-\beta_k\, x}}{(\alpha_k - 1)!}. \quad (40)$$

In this case a Semi-Markov kernel is

$$\mathbf{Q}(t) = \begin{bmatrix} Q_{00}(t) & 0 & 0 & 0 \\ Q_{10}(t) & 0 & 0 & 0 \\ Q_{20}(t) & Q_{21}(t) & 0 & 0 \\ Q_{30}(t) & 0 & Q_{32}(t) & 0 \end{bmatrix}. \quad (41)$$

Let us notice that this matrix is equal to the matrix (36) from the example 1 with $Q_{31}(t) = 0$. Therefore we can apply equalities (37), (38) and (39) to calculate components of multi-state reliability function. Finally we obtain Laplace transforms:

$$\tilde{R}_{3[1]}(s) = \frac{\tilde{u}_3(s)}{s}, \\ \tilde{u}_3(s) = 1 - \tilde{q}_{30}(s) - \tilde{q}_{20}(s)\tilde{q}_{32}(s) - \tilde{q}_{10}(s)\tilde{q}_{21}(s)\tilde{q}_{32}(s), \quad (42)$$

$$\tilde{R}_{3[2]}(s) = \frac{\tilde{u}_2(s)}{s}, \quad (43) \\ \tilde{u}_2(s) = 1 - \tilde{q}_{30}(s) - \tilde{q}_{20}(s)\tilde{q}_{32}(s) - \tilde{q}_{21}(s)\tilde{q}_{32}(s)$$

and

$$\tilde{R}_{3[3]}(s) = \frac{1 - \tilde{q}_{30} - \tilde{q}_{32}(s)}{s}, \quad (44)$$

where

$$\tilde{q}_{10}(s) = \frac{\beta_1^2}{(s+\beta_1+\lambda_1)^2} + \frac{\lambda_1(s+2\beta_1+\lambda_1)}{(s+\beta_1+\lambda_1)^2},$$
$$\tilde{q}_{21}(s) = \frac{\beta_2^2}{(s+\beta_2+\lambda_2)^2}, \quad \tilde{q}_{20}(s) = \frac{\lambda_2(s+2\beta_2+\lambda_2)}{(s+\beta_2+\lambda_2)^2}, \quad (45)$$
$$\tilde{q}_{32}(s) = \frac{\beta_3^2}{(s+\beta_3+\lambda_3)^2}, \quad \tilde{q}_{30}(s) = \frac{\lambda_3(s+2\beta_3+\lambda_3)}{(s+\beta_3+\lambda_3)^2}.$$

For a numerical example we take

$$\begin{aligned} &\alpha_1 = 2, \quad \beta_1 = 0.04, \quad \lambda_1 = 0.004, \\ &\alpha_2 = 2, \quad \beta_2 = 0.03, \quad \lambda_2 = 0.002, \\ &\alpha_3 = 2, \quad \beta_3 = 0.02, \quad \lambda_3 = 0.001. \end{aligned} \quad (46)$$

Substituting the functions (45) with parameters (46) to equations (42), (43) and (44) we obtain

$$\tilde{R}_{3[1]}(s) = \frac{\tilde{w}_1(s)}{(0.021+s)^2(0.032+s)^2(0.044+s)^2}$$

where

$$\tilde{w}_1(s) = 1.59533\ 10^{-7} + 0.000014s + 0.000626s^2 + 0.015224s^3 + 0.193s^4 + s^5,$$

and

$$\tilde{R}_{3[2]}(s) = \frac{0.000066784 + 0.004048s + 0.105s^2 + s^3}{(0.021+s)^2(0.032+s)^2},$$

$$\tilde{R}_{3[3]}(s) = \frac{0.041+s}{(0.021+s)^2}.$$

As the inverse Laplace transforms we obtain reliability functions

$$R_{3[1]}(t) = 52.6698e^{-0.044t} + 58.277e^{-0.032t} - 109.947e^{-0.021t} + 0.189036e^{-0.044t}t + 1.17355e^{-0.032t}t + 0.509862e^{-0.021t}t,$$

$$R_{3[2]}(t) = 21.3373e^{-0.032t} - 20.3373e^{-0.021t} + 0.0991736e^{-0.032t}t + 0.155537e^{-0.021t}t,$$

$$R_{3[3]}(t) = e^{-0.021t}(1 + 0.02\,t).$$

These reliability functions are shown in *Figure 4.*

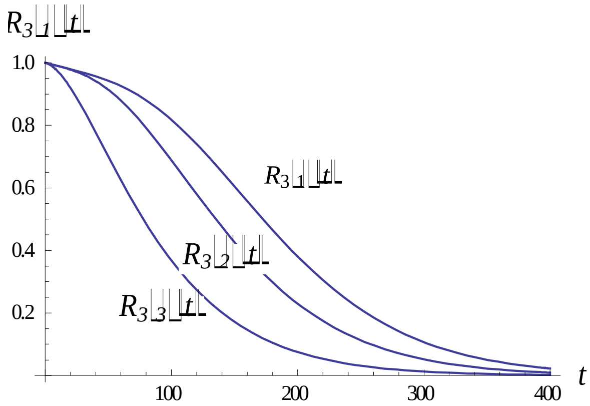

Figure 4. Components of Multi-State Reliability function

The corresponding expectations, second moments and standard deviations of l level system lifetime we calculate using formula's

$$m_1[l] = [E[T_{[l]}|X(0)=3] = \lim_{s\to 0} \tilde{R}_{3[l]}(s), \quad l = 1,2,3,$$
$$m_2[l] = E[T_{[l]}^2|X(0)=3] = -2\lim_{s\to 0}[\tilde{R}'_{3[l]}(s)],$$
$$\sigma[l] = \sqrt{m_2[l] - [m_1[l]]^2}.$$

For the given parameters we get

$$\begin{aligned} &m_1[1] = 182.48, \quad m_1[2] = 147.89, \quad m_1[3] = 92.97 \\ &m_2[1] = 41960.1, \quad m_2[2] = 28727.2, \quad m_2[3] = 13173.5 \\ &\sigma[1] = 93.06, \quad \sigma[2] = 82.80, \quad \sigma[3] = 67.30. \end{aligned}$$

5 . Inverse problem for simple damage exponential model

We suppose that there are possible the state changes only from k to $k-1$, $k = 1,2,\ldots,n$ with the positive probabilities. Now, the stochastic process $\{X(t):\ t \geqslant 0\}$, describing reliability state changes of the system, is the semi-Markov process with a state space $S = \{0,1,\ldots,n\}$ and a kernel

$$\mathbf{Q}(t) = \begin{bmatrix} Q_{00}(t) & 0 & 0 & \cdots & 0 \\ Q_{10}(t) & 0 & 0 & \cdots & 0 \\ 0 & Q_{21}(t) & 0 & \cdots & 0 \\ 0 & 0 & Q_{32}(t) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & Q_{n\,n-1}(t) & 0 \end{bmatrix}. \quad (47)$$

For simplicity we assume $n = 3$. From equations (42), (43), (44) we obtain the Laplace transforms of the multi-state reliability function components.

$$\tilde{R}_{3[1]}(s) = \frac{1 - \tilde{q}_{10}(s)\tilde{q}_{21}(s)\tilde{q}_{32}(s)}{s}, \quad (48)$$

$$\tilde{R}_{3[2]}(s) = \frac{1 - \tilde{q}_{21}(s)\tilde{q}_{32}(s)}{s}, \quad (49)$$

$$\tilde{R}_{3[3]}(s) = \frac{1 - \tilde{q}_{32}(s)}{s}. \quad (50)$$

The Mutli-State Reliability Function is called exponential if all of its component (except of $R_{n[0]}(t)$) are exponential functions [6], [13], [14]. In above presented model it means that

$$\tilde{R}_{3[l]}(s) = \frac{1}{s+\lambda_l}, \quad l = 1,2,3.$$

We set the following problem. Find elements

$$Q_{k,k-1}(t),\ k = 1,2,3$$

of the semi-Markov kernel. For calculating these functions we have to solve a following system of equations

$$\frac{1}{s+\lambda_1} = \frac{1 - \tilde{q}_{10}(s)\tilde{q}_{21}(s)\tilde{q}_{32}(s)}{s}, \quad (51)$$

$$\frac{1}{s+\lambda_2} = \frac{1 - \tilde{q}_{21}(s)\tilde{q}_{32}(s)}{s}, \quad (52)$$

$$\frac{1}{s+\lambda_3} = \frac{1 - \tilde{q}_{32}(s)}{s}, \quad (53)$$

where

$$0 < \lambda_1 < \lambda_2 < \lambda_3.$$

A solution of this system equations are Laplace transforms

$$\tilde{q}_{10}(s) = \tfrac{\lambda_1(s+\lambda_2)}{(s+\lambda_1)\lambda_2}.$$

$$\tilde{q}_{21}(s) = \tfrac{\lambda_2(s+\lambda_3)}{(s+\lambda_2)\lambda_3},$$

$$\tilde{q}_{32}(s) = \tfrac{\lambda_3}{s+\lambda_3}.$$

We obtain the functions $Q_{k\,k-1}(t),\ k = 1,2,3$ as the inverse Laplace transforms of

$$\tilde{Q}_{k\,k-1}(s) = \frac{\tilde{q}_{k\,k-1}(s)}{s}, \quad k = 1,2,3$$

Since we obtain

$$Q_{10}(t) = 1 - \left(1 - \tfrac{\lambda_1}{\lambda_2}\right) e^{-\lambda_1 t}, \quad t \geqslant 0,$$

$$Q_{21}(t) = 1 - \left(1 - \tfrac{\lambda_2}{\lambda_3}\right) e^{-\lambda_2 t}, \quad t \geqslant 0,$$

$$Q_{32}(t) = 1 - e^{-\lambda_3 t}, \quad t \geqslant 0.$$

Therefore the CDF of the waiting times T_i (7) for the states *i*=1 and *i* = 2 are

$$G_1(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ 1 - \left(1 - \tfrac{\lambda_1}{\lambda_2}\right) e^{-\lambda_1 t} & \text{for} \quad t \geqslant 0 \end{cases}$$

$$G_2(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ 1 - \left(1 - \tfrac{\lambda_2}{\lambda_3}\right) e^{-\lambda_2 t} & \text{for} \quad t \geqslant 0 \end{cases}$$

and for *I* =3 we have

$$G_3(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ 1 - e^{-\lambda_3 t} & \text{for} \quad t \geqslant 0 \end{cases}$$

Theorem 1.
For multi-state exponential reliability function

$$\mathbf{R}(t) = \left[1,\, e^{-\lambda_1 t},\, e^{-\lambda_2 t}, \ldots,\, e^{-\lambda_n t}\right],$$

where

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n,$$

the CDF of the waiting time T_k of the semi-Markov process defined by the kernel (56) is

$$G_k(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ 1 - \left(1 - \tfrac{\lambda_k}{\lambda_{k+1}}\right) e^{-\lambda_k t} & \text{for} \quad t \geqslant 0 \end{cases}$$

for $k = 1, \ldots, n-1$ and

$$G_n(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ 1 - e^{-\lambda_n t} & \text{for} \quad t \geqslant 0 \end{cases} \qquad \textit{for} \quad k{=}n$$

Proof: The same as for *n*=3.

Since, the probability distribution of the random variables T_k, $k = 1, 2, \ldots, n-1$ is a mixture of a discrete and absolutely continuous distribution.

$$G_k(t) = p\, G_k^{(d)}(t) + q\, G_k^{(c)}(t), \quad k = 1, \ldots n-1$$

where

$$p = \frac{\lambda_k}{\lambda_{k+1}}, \quad q = 1 - \frac{\lambda_k}{\lambda_{k+1}},$$

$$G_k^{(d)}(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ 1 & \text{for} \quad t \geqslant 0 \end{cases},$$

$$G_n^{(c)}(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ 1 - e^{-\lambda_k t} & \text{for} \quad t \geqslant 0 \end{cases}.$$

It follows from the above presented theorem that

$$P(T_k = 0) = \frac{\lambda_k}{\lambda_{k+1}}, \quad k = 1, \ldots, n-1.$$

It means, that there is possible a sequence of state changes $(n, n-1, \ldots, 1, 0)$ with the waiting times $(T_n > 0,\ T_{n-1} = 0, \ldots, T_1 = 0)$.
The probability of the sequence of these events is

$$P(T_n > 0,\ T_{n-1} = 0, \ldots, T_1 = 0) =$$

$$= \frac{\lambda_{n-1}}{\lambda_n} \frac{\lambda_{n-2}}{\lambda_{n-1}} \cdots \frac{\lambda_1}{\lambda_2} = \frac{\lambda_1}{\lambda_n}$$

In this case a value of *n*-level time to failure is

$$T_{n[n]} = t_n + 0 + \ldots + 0 = t_n,$$

where t_n is the value of the random variable T_n.

6 . Conclusion

Constructing muti-state semi-Markov models allow us to find the eliability characteristics and parameters of un-repairable systems. The multi-state reliability functions and corresponding expectations, second moments and standard deviations are calculated for presented systems. The solutions of the equations, which follow from the semi-Markov processes theory, are obtained by using the Laplace transformation. Some interesting conclusions follow from presented Theorem 1 concerning the multi state exponential reliability function. It is possible a sequence of the state changes $(n, n-1, \ldots, 1, 0)$ with waiting times

$$T_n > 0, T_{n-1} = 0, \ldots, T_1 = 0.$$

The probability of these events sequence is

$$P(T_n > 0,\ T_{n-1} = 0, \ldots, T_1 = 0) = \frac{\lambda_1}{\lambda_n}.$$

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