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**Reliability of large systems**

**Keywords**

reliability, large system, asymptotic approach, limit reliability function

**Abstract**

The paper is concerned with the application of limit reliability functions to the reliability evaluation of large systems. Two-state large non-repaired systems composed of independent components are considered. The asymptotic approach to the system reliability investigation and the system limit reliability function are defined. Two-state homogeneous series, parallel and series-parallel systems are defined and their exact reliability functions are determined. The classes of limit reliability functions of these systems are presented. The article contains an exemplary application of the presented facts to the reliability evaluation of large technical systems. The accuracy of this evaluation is illustrated. Brief review and the latest references on limit reliability functions of two-state and multi-state homogeneous and non-homogeneous series, parallel, “*m* out of *n*”, series-parallel, parallel-series, series-“*m* out of *n*”, series- “*m* out of *n*” and hierarchical systems are presented as well.

**1. Introduction**

Many technical systems belong to the class of complex systems as a result of the large number of components they are built of and their complicated operating processes. As a rule these are series systems composed of large number of components. Sometimes the series systems have either components or subsystems reserved and then they become parallel-series or series-parallel reliability structures. We meet large series systems, for instance, in piping transportation of water, gas, oil and various chemical substances. Large systems of these kinds are also used in electrical energy distribution. A city bus transportation system composed of a number of communication lines each serviced by one bus may be a model series system, if we treat it as not failed, when all its lines are able to transport passengers. If the communication lines have at their disposal several buses we may consider it as either a parallel-series system or an “*m* out of *n*” system. The simplest example of a parallel system or an “*m* out of *n*” system may be an electrical cable composed of a number of wires, which are its basic components, whereas the transmitting electrical network may be either a parallel-series system or an “*m* out of *n*”-series system. Large systems of these types are also used in telecommunication, in rope transportation and in transport using belt conveyers and elevators. Rope transportation systems like port elevators and ship-rope elevators used in shipyards during ship docking are model examples of series-parallel and parallel-series systems.

In the case of large systems, the determination of the exact reliability functions of the systems leads us to complicated formulae that are often useless for reliability practitioners. One of the important techniques in this situation is the asymptotic approach to system reliability evaluation. In this approach, instead of the preliminary complex formula for the system reliability function, after assuming that the number of system components tends to infinity and finding the limit reliability of the system, we obtain its simplified form.

The mathematical methods used in the asymptotic approach to the system reliability analysis of large systems are based on limit theorems on order statistics distributions, considered in very wide literature, for instance in [3]-[4], [6], [9]. These theorems have generated the investigation concerned with limit reliability functions of the systems composed of two-state components. The main and fundamental results on this subject that determine the three-element classes of limit reliability functions for homogeneous series systems and for homogeneous parallel systems have been established by Gniedenko in [5]. These results are also presented, sometimes with different proofs, for instance in subsequent works [1], [7]. The generalizations of these results for homogeneous “*m* out of *n*” systems have been formulated and proved by Smirnow in [10], where the seven-element class of possible limit reliability functions for these systems has been fixed. As it has been done for homogeneous series and parallel systems classes of limit reliability functions have been fixed by Chernoff and Teicher in [2] for homogeneous series-parallel and parallel-series systems. Their results were concerned with so-called “quadratic” systems only. They have fixed limit reliability functions for the homogeneous series-parallel systems with the number of series subsystems equal to the number of components in these subsystems, and for the homogeneous parallel-series systems with the number of parallel subsystems equal to the number of components in these subsystems. Kolowrocki has generalized their results for non-“quadratic” and non-homogeneous series-parallel and parallel-series systems in [7]. These all results may also be found for instance in [8].

All the results so far described have been obtained under the linear normalization of the system lifetimes. The paper contains the results described above and comments on their newest generalizations recently presented in [8].

*En*

*E*2

*E*1

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.

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 . . .

*E*1

*E*2

*En*

#### 2. Reliability of two-state systems

We assume that

 *Ei*, *i* = 1,2,...,*n*, *n* ∈ *N*,

are two-state components of the system having reliability functions

 *Ri*(*t*) = *P*(*Ti* > *t*), 

where

 *Ti*, *i* = 1,2,...,*n*,

are independent random variables representing the lifetimes of components *Ei* with distribution functions

 *Fi*(*t*) = *P*(*Ti* ≤ *t*), 

The simplest two-state reliability structures are series and parallel systems. We define these systems first.

*Definition 1*. We call a two-state system series if its lifetime *T* is given by

 *T* = 

###### The scheme of a series system is given in *Figure 1*.

###### *Figure 1*. The scheme of a series system

*Definition 1* means that the series system is not failed if and only if all its components are not failed, and therefore its reliability function is given by

****** = ,  (1)

*Definition 2*. We call a two-state system parallel if its lifetime *T* is given by

 *T* = 

The scheme of a parallel system is given in *Figure 2*.

*Figure 2*. The scheme of a parallel system

*Definition 2* means that the parallel system is failed if and only if all its components are failed and therefore its reliability function is given by

 ***R****n*(*t*) = 1 ,  (2)

Another basic, a bit more complex, two-state reliability structure is a series-parallel system. To define it, we assume that

 *Eij*, *i* = 1,2,...,*kn*, *j* = 1,2,...,*li*, *kn*, *l*1, *l*2,...,∈ *N*,

are two-state components of the system having reliability functions

 *Rij*(*t*) = *P*(*Tij* > *t*), 

where

 *Tij*, *i* = 1,2,...,*kn*, *j* = 1,2,...,*li*,

are independent random variables representing the lifetimes of components *Eij* with distribution functions

 *Fij*(*t*) = *P*(*Tij* ≤ *t*), 

*Definition 3*. We call a two-state system series-parallel if its lifetime *T* is given by

 *T* = 

By joining the formulae (1) and (2) for the reliability functions of two-state series and parallel systems it is easy to conclude that the reliability function of the two-state series-parallel system is given by

 ***R*** = , (3)



where *kn* is the number of series subsystems linked in parallel and *li* are the numbers of components in the series subsystems.

*Definition 4*. We call a two-state series-parallel system regular if

 *l*1 = *l*2 = . . . = = *ln*, *ln* ∈ *N*,

i.e. if the numbers of components in its series subsystems are equal.

###### The scheme of a regular series-parallel system is given in *Figure 3*.



*Figure 3*. The scheme of a regular series parallel system

*Definition 5*. We call a two-state system homogeneous if its component lifetimes have an identical distribution function ** i.e. if its components have the same reliability function

 ** 

The above definition and equations (1)-(3) result in the simplified formulae for the reliability functions of the homogeneous systems stated in the following corollary.

*Corollary 1*. The reliability function of the homogeneous two-state system is given by

######

- for a series system

 ****** = [*R*(*t*)]*n*,  (4)

- for a parallel system

 ***R****n*(*t*) =   (5)

- for a regular series-parallel system

 ***R*** =   (6)

**3. Asymptotic approach to system reliability**

The asymptotic approach to the reliability of two-state systems depends on the investigation of limit distributions of a standardized random variable

 

where *T* is the lifetime of a system and *an* > 0,  are suitably chosen numbers called normalizing constants.

Since



***R****n*(*ant* + *bn*),

where ***R****n*(*t*) is a reliability function of a system composed of *n* components, then the following definition becomes natural.

*Definition 6*. We call a reliability function ***ℜ***(*t*) the limit reliability function of a system having a reliability function ***R****n*(*t*) if there exist normalizing constants *an* > 0, *bn*∈ (-∞, ∞) such that

 ***R****n*(*ant* + *bn*) = ***ℜ***(*t*) for *t* ∈ *C****ℜ***,

where *C****ℜ***  is the set of continuity points of ***ℜ***(*t*).

Thus, if the asymptotic reliability function ***ℜ***(*t*) of a system is known, then for sufficiently large *n*, the approximate formula

 ***R****n*(*t*) ≅ ***ℜ***(/*an*),  (7)

may be used instead of the system exact reliability function ***R****n*(t).

**3.1. Reliability of large two-state series systems**

The investigations of limit reliability functions of homogeneous two-state series systems are based on the following auxiliary theorem.

*Lemma 1*. If

 (i)  is a non-degenerate reliability function,

 (ii)****** is the reliability function of a homogeneous two-state series system defined by (4),

(iii) 

then

 ******(*ant* + *bn*) = for *t* ∈ 

if and only if

 *nF*(*ant* + *bn*) =** for *t* ∈ **

*Proof*. The proof may be found in [1], [5], [7].

*Lemma 1* is an essential tool in finding limit reliability functions of two-state series systems. It also is the basis for fixing the class of all possible limit reliability functions of these systems. This class is determined by the following theorem.

*Theorem 1*. The only non-degenerate limit reliability functions of the homogeneous two-state series system are:

 =  for *t* < 0,

 = 0 for *t* ≥ 0, *α* > 0;

  = 1 for *t* < 0,

  =  for *t* ≥ 0, *α* > 0;

 =  for 

*Proof*. The proof may be found in [1], [5], [7].

**3.2. Reliability of large two-state parallel**

**systems**

The class of limit reliability functions for homogeneous two-state parallel systems may be determined on the basis of the following auxiliary theorem.

*Lemma 2*. If

 (i) ***ℜ***(*t*) =  is a non-degenerate reliability function,

(ii) ***R****n*(*t*) is the reliability function of a homogeneous two-state parallel system defined by (5),

 (iii) 

then

 ***R****n*(*ant* + *bn*) = ***ℜ***(*t*) for *t* ∈,

if and only if

 *nR*(*ant* + *bn*) = *V*(*t*) for *t* ∈.

*Proof*. The proof may be found in [1], [5], [7].

By applying *Lemma 2* it is possible to fix the class of limit reliability functions for homogeneous two-state parallel systems. However, it is easier to obtain this result using the duality property of parallel and series systems expressed in the relationship

 ***R****n*(*t*) = ****** for 

that results in the following lemma, [1], [5], [7]-[8].

*Lemma 3*. If  is the limit reliability function of a homogeneous two-state series system with reliability functions of particular components  then

 ***ℜ***(*t*) = 1  for *t* ∈ 

is the limit reliability function of a homogeneous two-state parallel system with reliability functions of particular components

 ****** for 

At the same time, if  is a pair of normalizing constants in the first case, then  is such a pair in the second case.

The application of *Lemma 3* and *Theorem 1* yields the following result.

*Theorem 2*. The only non-degenerate limit reliability functions of the homogeneous parallel system are:

 ***ℜ***1(*t*) = 1 for *t* ≤ 0,

 ***ℜ***1(*t*) = 1 − exp[−*t*−*α*] for *t* > 0, *α* > 0;

 ***ℜ***2(*t*) = 1 − exp[−(−*t*)*α*] for *t* < 0,

 ***ℜ***2(*t*) = 0 for *t* ≥ 0, *α* > 0;

 ***ℜ***3(*t*) = 1 − exp[−exp[−*t*]] for *t* ∈ (−∞,∞).

*Proof*. The proof may be found in [1], [5], [7].

**3.3. Reliability evaluation of large two-state series-parallel systems**

The proofs of the theorems on limit reliability functions for homogeneous regular series-parallel systems and methods of finding such functions for individual systems are based on the following essential lemmas.

*Lemma 4*. If

 (i) *kn* → ∞,

 (ii) ***ℜ***(*t*) = 1 − exp[−*V*(*t*)] is a non-degenerate reliability function,

(iii) ***R***(*t*) is the reliability function of a homogeneous regular two-state series-parallel system defined by (6),

 (iv) *an* > 0, *bn* ∈ (−∞,∞),

then

 ** *R***(*ant* + *bn*) = ***ℜ***(*t*) for *t* ∈,

if and only if

 *****kn*[*R*(*ant* + *bn*) = *V*(*t*) for *t* ∈.

*Proof*. The proof may be found in [7].

#### *Lemma 5*. If

#####

 (i) *kn* → *k*, , *ln* → ∞,

 (ii) ***ℜ***(*t*) is a non-degenerate reliability function,

(iii) ***R***(*t*) is the reliability function of a homogeneous regular two-state series-parallel system defined by (6),

 (iv) *an* > 0, *bn* ∈ (−∞,∞),

then

 ** *R***(*ant* + *bn*) = ***ℜ***(*t*) for *t* ∈,

if and only if

 ****[*R*(*ant* + *bn*) = ***ℜ***0(*t*) for *t* ∈,

where ***ℜ***0(*t*) is a non-degenerate reliability function and moreover

 ***ℜ***(*t*) = 1 − [1 − ***ℜ***0(*t*)]*k*for *t* ∈ (−∞,∞).

*Proof*. The proof may be found in [7].

The types of limit reliability functions of a series-parallel system depend on the system shape [7], i.e. on the relationships between the number *kn* of its series subsystems linked in parallel and the number *ln* of components in its series subsystems. The results based on *Lemma 4* and *Lemma 5* may be formulated in the form of the following theorem.

*Theorem 3*. The only non-degenerate limit reliability functions of the homogeneous regular two-state series-parallel system are:

*Case 1*. *kn* = *n*, ⏐*ln* − *c* log *n*⏐ >> *s*, *s* > 0, *c* > 0.

 ***ℜ***1(*t*) = 1 for *t* ≤ 0,

 ***ℜ***1(*t*) = 1 − exp[] for *t* > 0, *α* > 0;

 ***ℜ***2(t) = 1 − exp[− (−*t*)*α*] for *t* < 0,

 ***ℜ***2(*t*) = 0 for *t* ≥ 0, *α* > 0;

 ***ℜ***3(t) = 1 − exp[−exp[−*t*]] for *t* ∈ (−∞,∞);

*Case 2*. *kn* = *n*, *ln* − *c* log *n* ≈ *s*, *s* ∈ (−∞,∞),

*c* > 0.

 ***ℜ***4(*t*) = 1 for *t* < 0,

 ***ℜ***4(*t*) = 1 - exp[−exp[−*tα* − *s*/*c*]] for *t* ≥ 0, *α* > 0;

 ***ℜ***5(*t*) = 1 − exp[−exp[(−*t*)*α* − *s*/*c*]] for *t* < 0,

 ***ℜ***5(*t*) = 0 for *t* ≥ 0, *α* > 0;

 ***ℜ***6(*t*) = 1 − exp[−exp[*β*(−*t*)*α* − *s*/*c*]] for *t* < 0,

 ***ℜ***6(t) = 1 − exp[−exp[−*t*α − *s*/*c*]] for t ≥ 0,

 *α* > 0, *β* > 0;

 ***ℜ***7(*t*) = 1 for *t* < *t*1,

 ***ℜ***7(*t*) = 1 − exp[−exp[−*s*/*c*]] for *t*1 ≤ *t* < *t*2,

 ***ℜ***7(*t*) = 0 for *t* ≥ *t*2, *t*1 < *t*2;

*Case 3*. *kn* → *k*, , *ln* → ∞.

 ***ℜ***8(*t*) = 1 − [1 − exp[]]*k* for *t* < 0,

 ***ℜ***8(*t*) = 0 for *t* ≥ 0, *α* > 0;

 ***ℜ***9(*t*) = 1 for *t* < 0,

 ***ℜ***9(*t*) = 1 − [1 − exp[−*tα*]]*k* for *t* ≥ 0, *α* > 0;

 ***ℜ***10(*t*) = 1 − [1 − exp[−exp *t*]]*k* for *t* ∈ (−∞,∞).

*Proof*. The proof may be found in [7].

Using the duality property of parallel-series and series-parallel systems similar to this given in *Lemma 3* for parallel and series systems it is possible to prove that the only limit reliability functions of the homogeneous regular two-state parallel-series system are

 ***ℜi***(-*t*) for *t* ∈  

**4. An example**

Using *Lemma 2*, it is possible to prove the following fact [8].

*Corollary 2*. If components of the homogeneous two-state parallel system have Weibull reliability functions

 *R*(*t*) = 1 for *t* < 0,

 *R*(*t*) = exp[−*β*] for *t* ≥ 0, *α* > 0, *β* > 0,

and

 *an* = *bn*/(*α*log *n*), *bn* = (log *n*/*β*)1/*α*,

then

 ***ℜ***3(*t*) = 1 − exp[−exp[−*t*]], *t* ∈ (−∞,∞),

is its limit reliability function.

# *Example 1* (*a steel rope, durability*). Let us consider a steel rope cable composed of 36 strands used in ship rope elevator and assume that it is not failed if at least one of its strands is not broken. Under this assumption we may consider the rope as a homogeneous parallel system composed of *n* = 36 basic components. The cross-section of the considered rope is presented in *Figure 4*.

*Figure* *4*. The cross-section of the steel rope

Further, assuming that the strands have Weibull reliability functions with parameters

 *α* = 2, *β* = (7.07)–6,

## by (5), the rope’s exact reliability function takes the form

 ***R***36(*t*) = 1 for *t* < 0,

 ***R***36(*t*) = 1 – [1 − exp[−(7.07)–6*t*2]36 for *t*  0.

Thus, according to *Corollary 2*, assuming

 *an* = (7.07)3/(2), *bn* = (7.07)3,

and applying (7), we arrive at the approximate formula for the rope reliability function of the form

 ***R***36(*t*) ≅ ***ℜ***3((*t* − *bn*)/*an*)

 = 1 − exp[−exp[−0.01071*t* + 7.167]]

 for *t* ∈ (−∞,∞).

The expected value of the rope lifetime *T* and its standard deviation, in months, calculated on the basis of the above approximate result and according to the formulae

 *E*[*T*] = *Can* + *bn*, 

where *C* ≅ 0.5772 is Euler’s constant, respectively are:

 *E*[*T*] ≅ 723, *σ* ≅ 120.

The values of the exact and approximate reliability functions of the rope are presented in *Table 1* and graphically in *Figure 5*.

*Table 1*. The values of the exact and approximate reliability functions of the steel rope

|  |  |  |  |
| --- | --- | --- | --- |
| *t* | ***R***36(*t*) | ***ℜ***3  |  = ***R***36 − ***ℜ***3 |
| 0 | 1.000 | 1.000 | 0.000 |
| 400 | 1.000 | 1.000 | 0.000 |
| 500 | 0.995 | 0.988 | −0.003 |
| 550 | 0.965 | 0.972 | −0.007 |
| 600 | 0.874 | 0.877 | −0.003 |
| 650 | 0.712 | 0.707 | 0.005 |
| 700 | 0.513 | 0.513 | 0.000 |
| 750 | 0.330 | 0.344 | −0.014 |
| 800 | 0.193 | 0.218 | −0.025 |
| 900 | 0.053 | 0.081 | −**0.028** |
| 1000 | 0.012 | 0.029 | −0.017 |
| 1100 | 0.002 | 0.010 | −0.008 |
| 1200 | 0.000 | 0.003 | −0.003 |

The differences between them are not large, which means that the mistakes in replacing the exact rope reliability function by its approximate form are practically not significant.

***R***36(*t*), ***ℜ***3((*t* − *bn*)/*an*)

1

0.8

0.6

0.4

0.2

0

400

800

1000

*t*

600

200

### Figure 5. The graphs of the exact and approximate reliability functions of the steel rope

# 5. Conclusion

Generalizations of the results on limit reliability functions of two-state homogeneous systems for these and other systems in case they are non-homogeneous, are mostly given in [7] and [8]. These results allow us to evaluate reliability characteristics of homogeneous and non-homogeneous series-parallel and parallel-series systems with regular reliability structures, i.e. systems composed of subsystems having the same numbers of components. However, this fact does not restrict the completeness of the performed analysis, since by conventional joining of a suitable number of components which do not fail, in series sub-systems of the non-regular series-parallel systems, leads us to the regular non-homogeneous series-parallel systems. Similarly, conventional joining of a suitable number of failed components in parallel subsystems of the non-regular parallel-series systems we get the regular non-homogeneous parallel-series systems. Thus the problem has been analyzed exhaustively.

More general and practically important complex systems composed of multi-state and degrading in time components are considered in wide literature, for instance in [11]. An especially important role they play in the evaluation of technical systems reliability and safety and their operating process effectiveness is described in [8] for large multi-state systems with degrading components. The most important results regarding generalizations of the results on limit reliability functions of two-state systems dependent on transferring them to series, parallel, “*m* out of *n*”, series-parallel and parallel-series multi-state systems with degrading components are given in [8]. Some practical applications of the asymptotic approach to the reliability evaluation of various technical systems are contained in [8] as well.

The results concerned with the asymptotic approach to system reliability analysis have become the basis for the investigation concerned with domains of attraction for the limit reliability functions of the considered systems. In a natural way they have led to investigation of the speed of convergence of the system reliability function sequences to their limit reliability functions. These results have also initiated the investigation of limit reliability functions of “*m* out of *n*”-series, series-“*m* out of *n*” systems, and systems with hierarchical reliability structures, as well as investigations on the problems of the system reliability improvement and optimization described briefly in [8].

The proposed method offers enough simplified formulae to allow significant simplifying of reliability evaluating and optimizing calculations.

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